CPEN 455: Deep Learning

Lecture 10: Autoencoders, Denoising Autoencoders, and Variational Autoencoders

Original slides by **Renjie Liao**
Presented by **Qihang Zhang**

University of British Columbia
Winter, Term 2, 2024

Outline

- Autoencoders
  - **Motivation & Overview**
  - Linear Autoencoders & PCA
  - Deep Autoencoders
- Denoising Autoencoders
- Variational Autoencoders
  - Motivation & Overview
  - Evidence Lower Bound (ELBO)
  - Models
  - Amortized Inference
  - Reparameterization Trick

Autoencoders (AEs)

- Autoencoders are feed-forward neural networks that reconstruct/predict the input

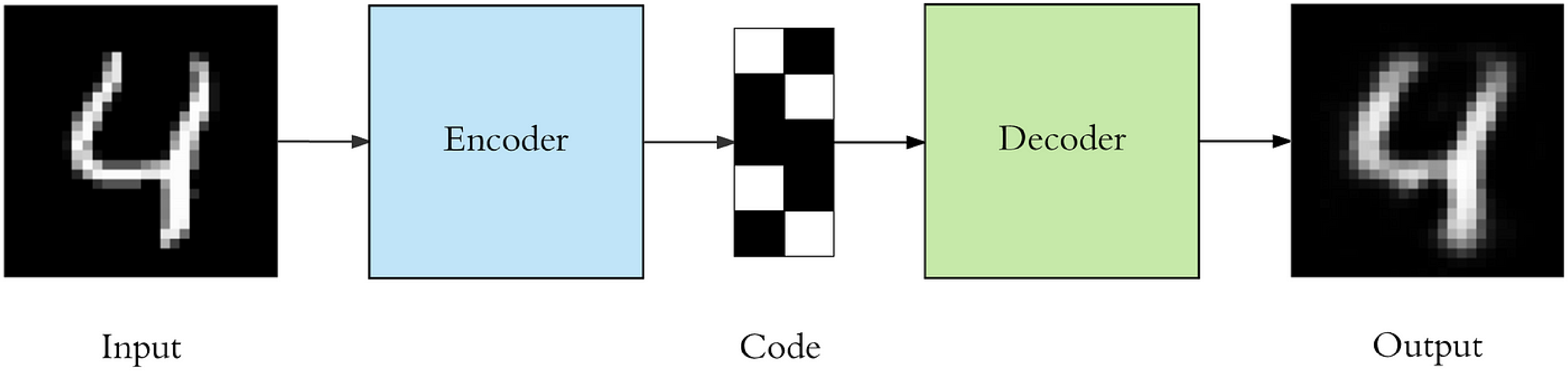
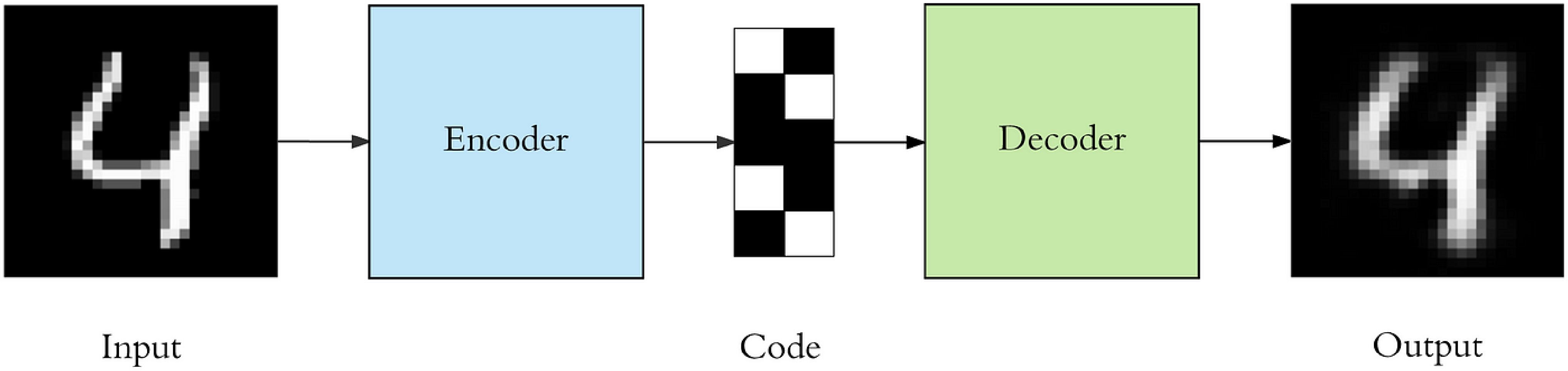


Image Credit: [1]

Autoencoders (AEs)

- Autoencoders are feed-forward neural networks that reconstruct/predict the input
- To make it non-trivial, we need a *bottleneck* (i.e. the dimension of code being much smaller compared to the input). (explain later).



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Autoencoders (AEs)

Why should we care?

- Dimension reduction

    e.g., visualizing high-dimension data

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- Dimension reduction

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- Unsupervised representation learning

  e.g., if we have abundant data without annotations, learned representations will potentially be useful for downstream tasks like classification and regression

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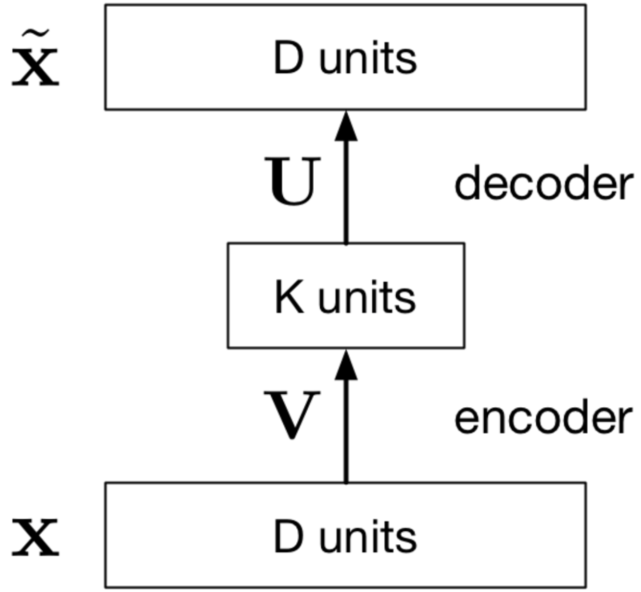
Linear Autoencoders

Simplest autoencoders: a single hidden layer with linear activations

We can train them by minimizing the mean squared errors (MSE):

$$\ell(\tilde{\mathbf{x}}, \mathbf{x}) = \|\tilde{\mathbf{x}} - \mathbf{x}\|_2^2$$

The network is $\tilde{\mathbf{x}} = UV\mathbf{x}$



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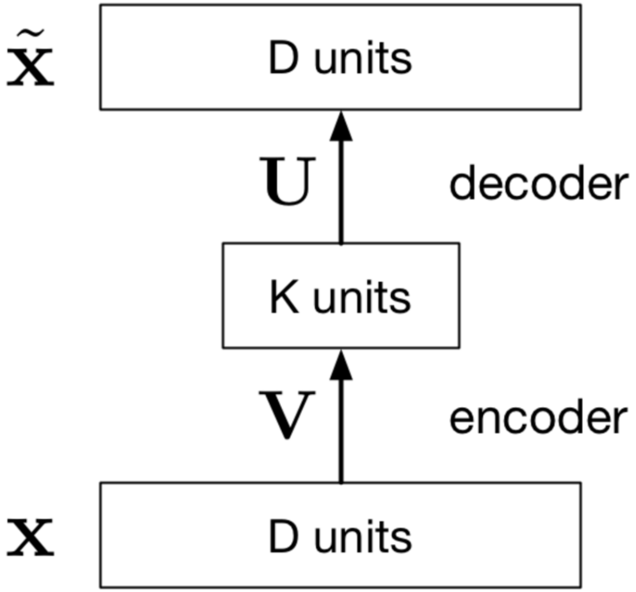
If $K \geq D$, one can choose U and V such that $UV = I$ (copying input)

Underdetermined system of equations, possibly having infinite solutions



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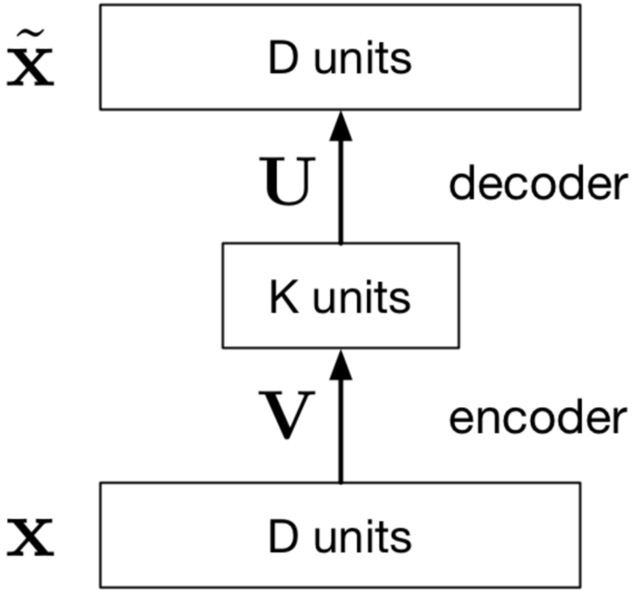
The reconstructed output lies in the column space of U, which is a K-dimensional subspace



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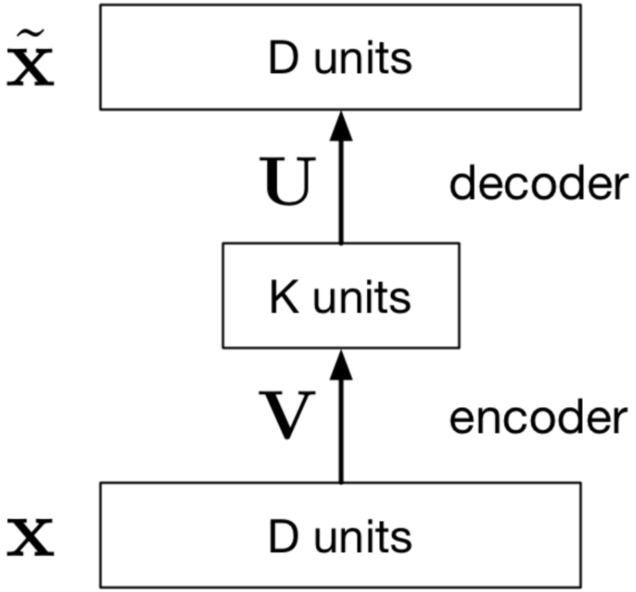

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$$\hat{x} = U^{D\times K} y = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_K] \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_K \end{pmatrix} = \sum_{i=1}^{K} y_i \vec{u}_i$$

$$y = V^{K\times D} x$$

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Linear Autoencoders & Principle Component Analysis

We know linear autoencoders map D-dimensional input to a K-dimensional subspace

What is the best possible K-dimensional mapping?

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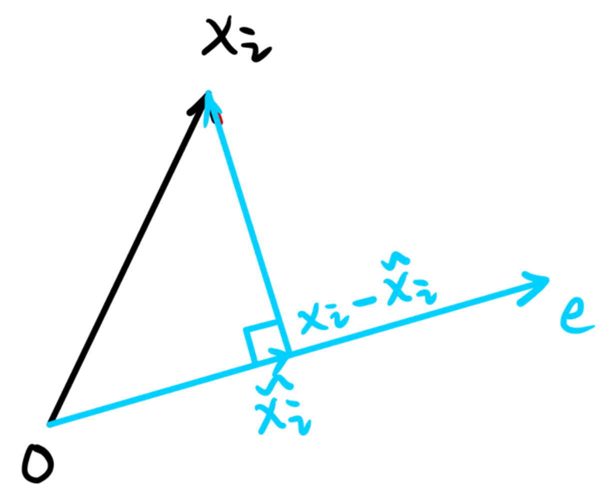

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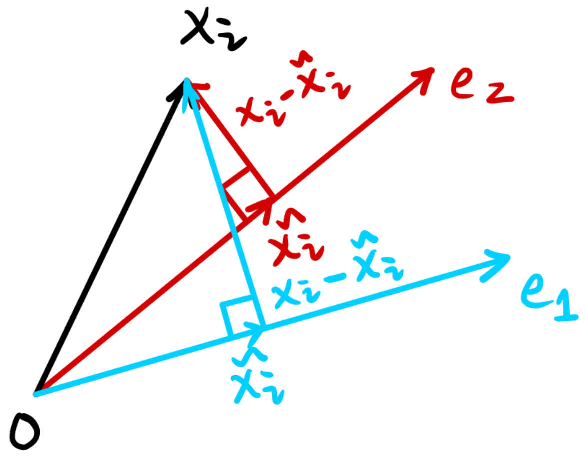

$$\|x_i\|^2 = \|\hat{x}_i\|^2 + \|x_i - \hat{x}_i\|^2$$

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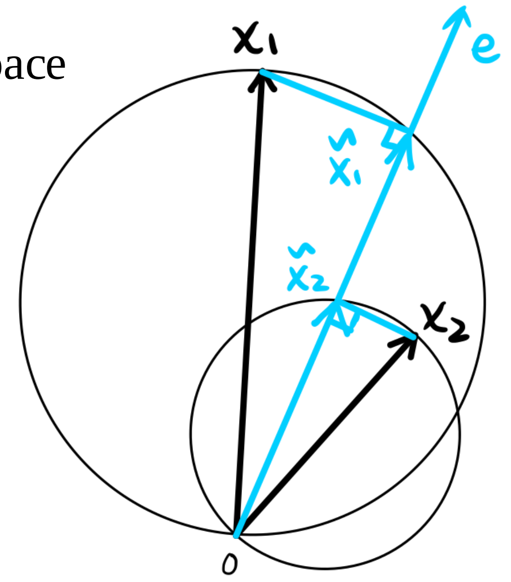

$$\frac{1}{2}\sum_{i=1}^{2}\|x_i\|^2 = \frac{1}{2}\sum_{i=1}^{2}\|\hat{x}_i\|^2 + \frac{1}{2}\sum_{i=1}^{2}\|x_i - \hat{x}_i\|^2$$

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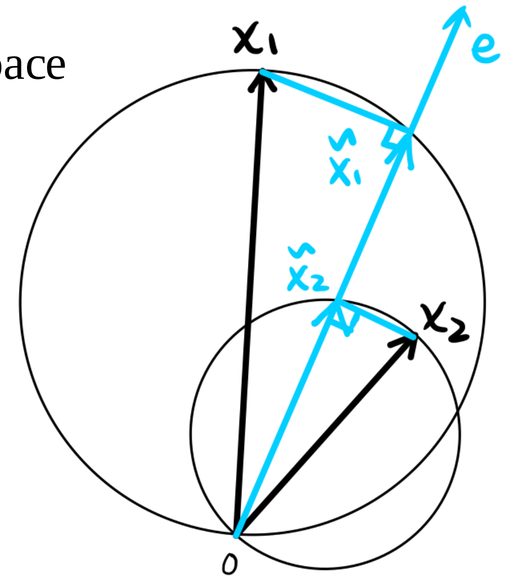
By Pythagorean Theorem, we have:

$$\underbrace{\frac{1}{N}\sum_{i=1}^{N}\|\mathbf{x}_i\|^2}_{\text{constant}} = \underbrace{\frac{1}{N}\sum_{i=1}^{N}\|\mathbf{x}_i - \tilde{\mathbf{x}}_i\|^2}_{\text{reconstruction error}} + \underbrace{\frac{1}{N}\sum_{i=1}^{N}\|\tilde{\mathbf{x}}_i\|^2}_{\text{projected variance}}$$



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Maximizing the projected variance is equivalent to minimizing the reconstruction error!

You can maximize the variance in closed-form via *principle component analysis (PCA)*!

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can exactly recover principled components!

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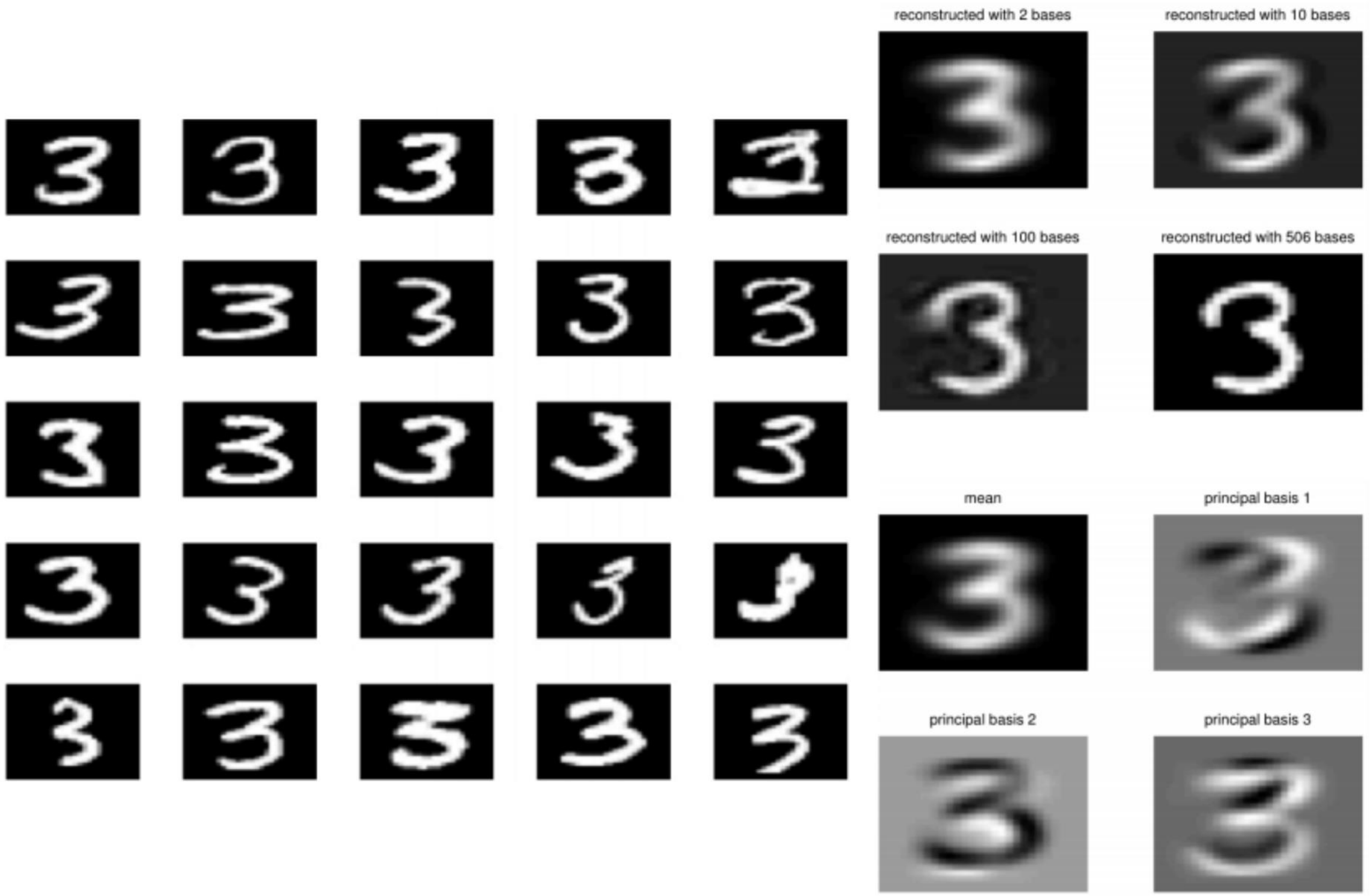
Principle components of faces ("Eigenfaces") from CBCL dataset:

Image Credit: [2]

Linear Autoencoders & Principle Component Analysis

Principle components of digits from MNIST dataset:



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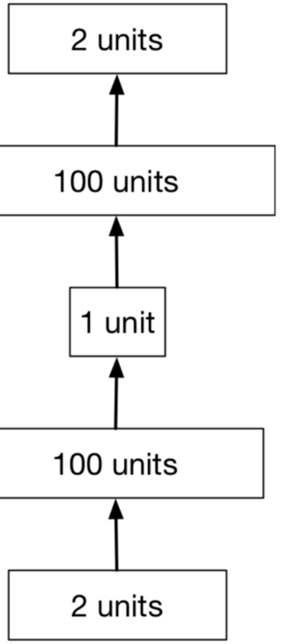
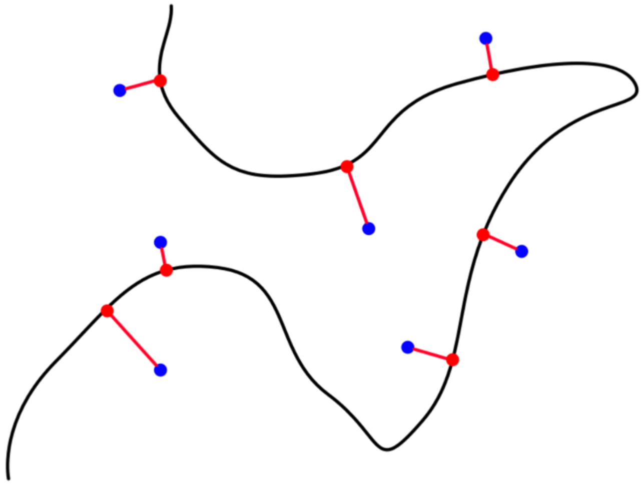
Deep Autoencoders

Deep autoencoders learn to project data onto a *manifold* instead of a subspace

This is a kind of *nonlinear dimension reduction*





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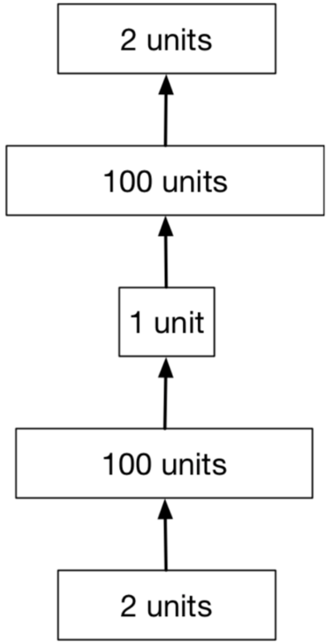

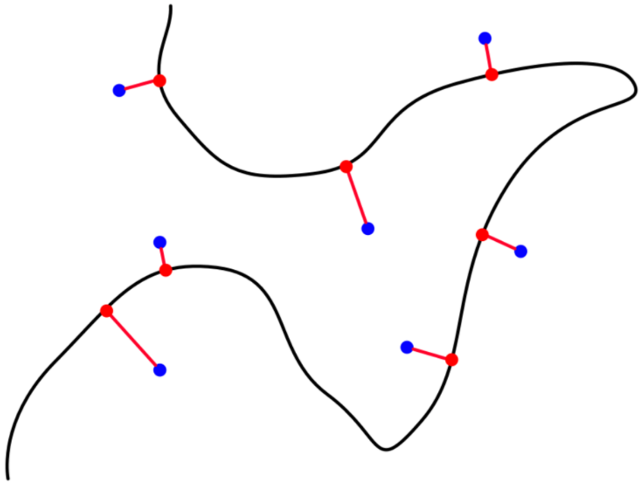

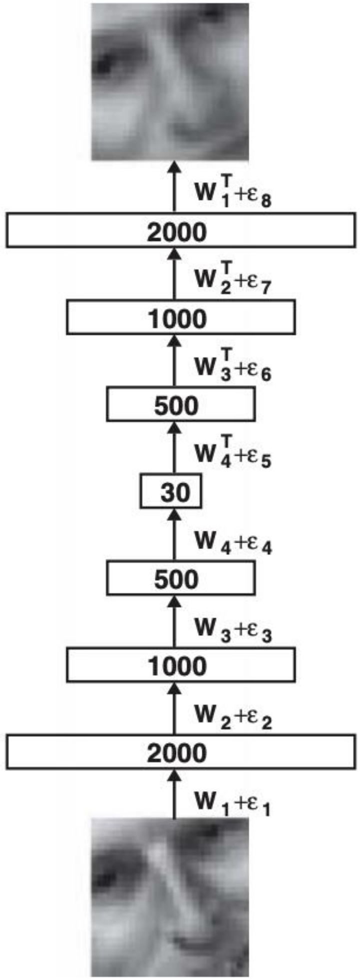

Image Credit: [2,4]

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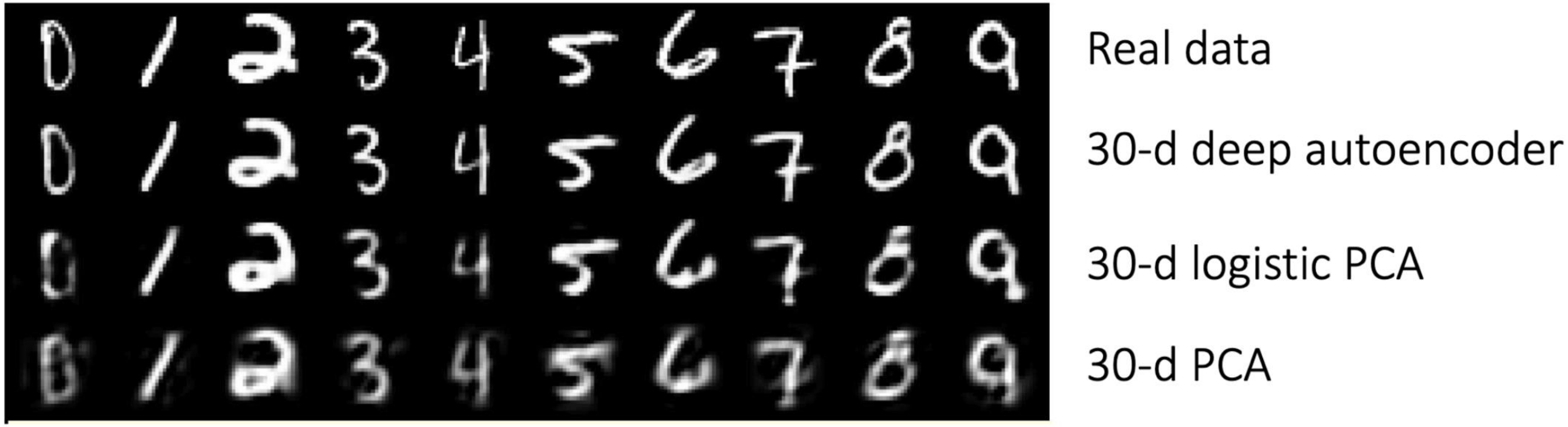
Deep autoencoders can learn more powerful codes/representations compared to linear ones (PCA)

Reconstructions with various methods on MNIST dataset:



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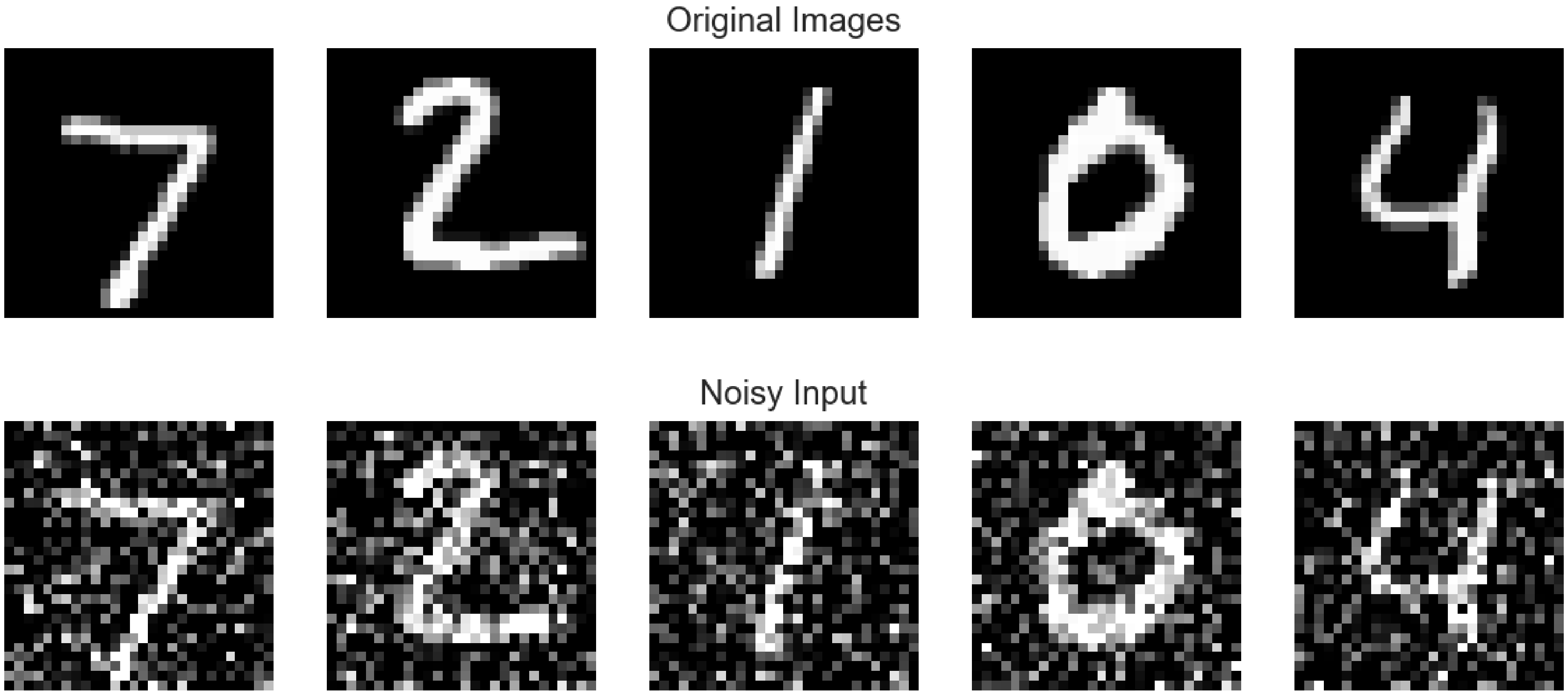

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Reconstructing input data is not the only way to learn useful representations in an unsupervised way.

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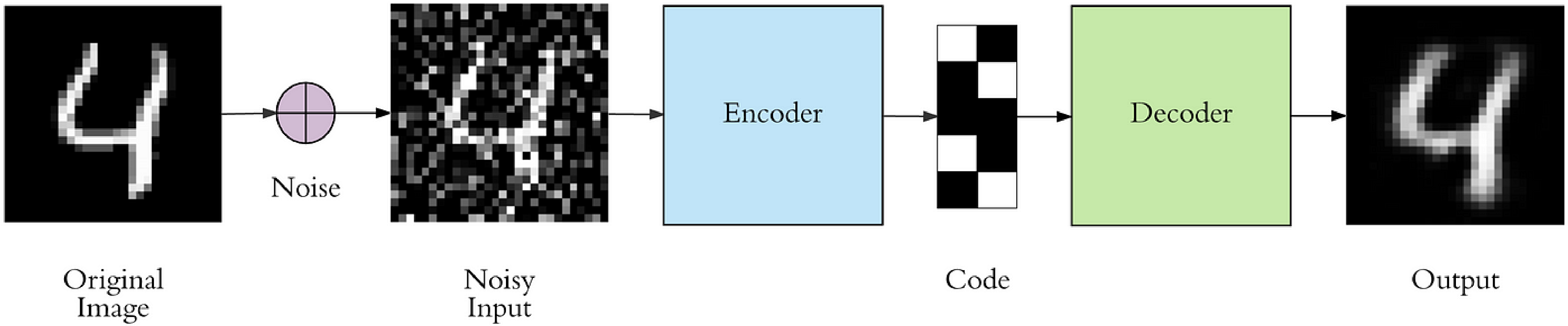
We add random noise (e.g., additive Gaussian) and force the neural network to learn useful representations so that *structures in images are preserved whereas noise is removed!*



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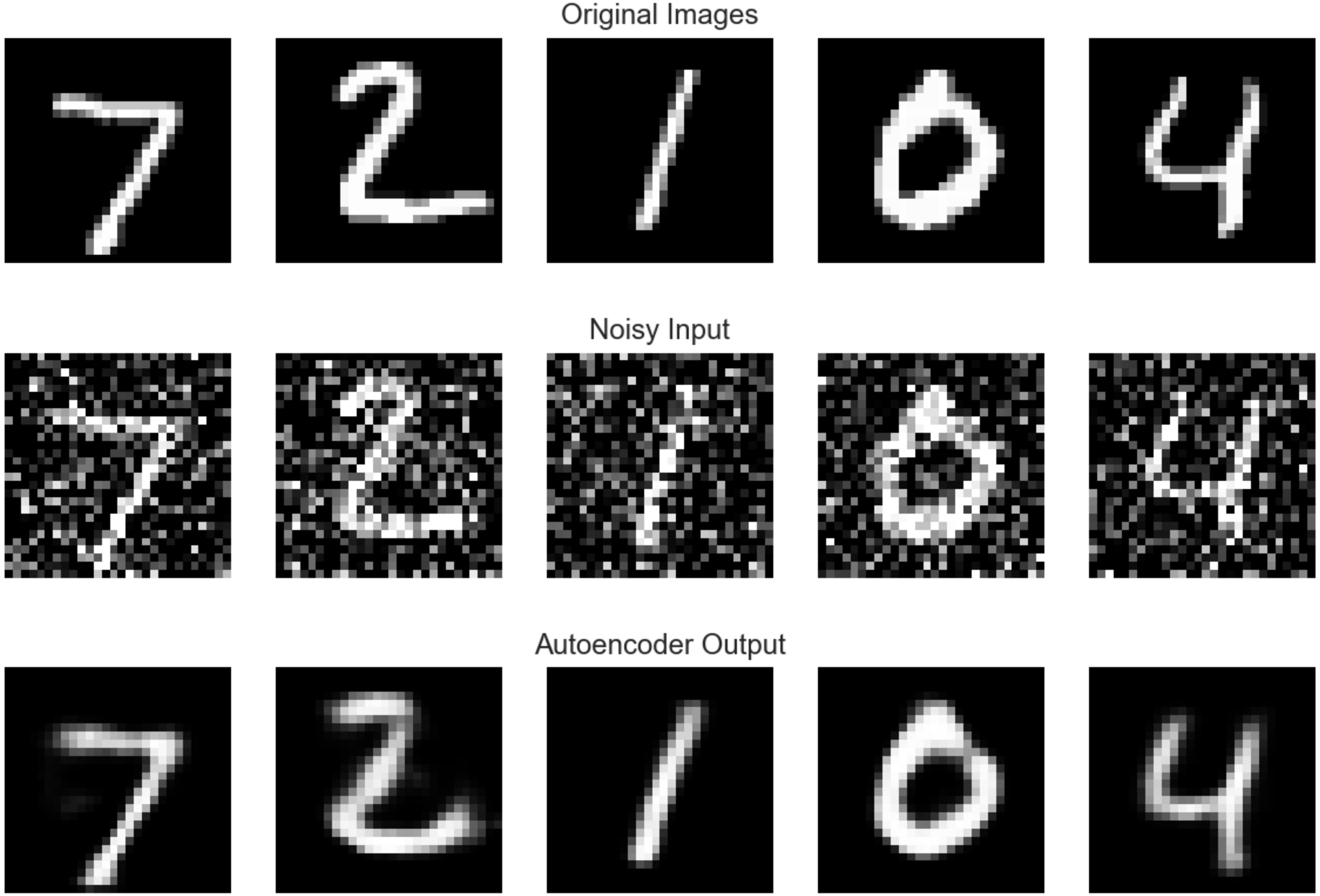
DAEs can do a great job in denoising:



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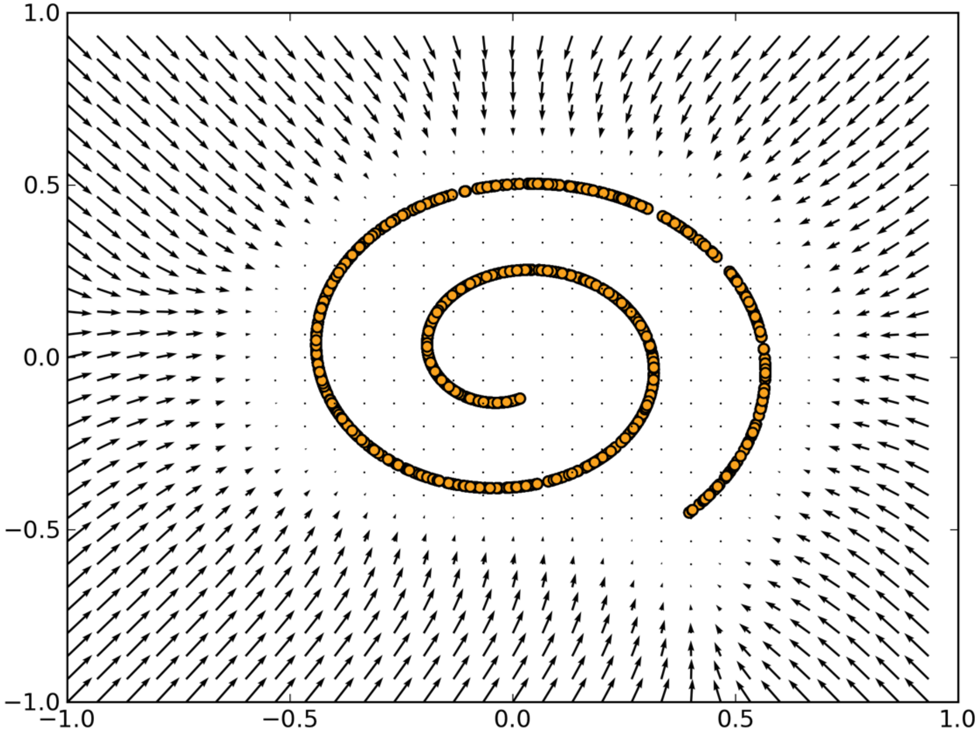
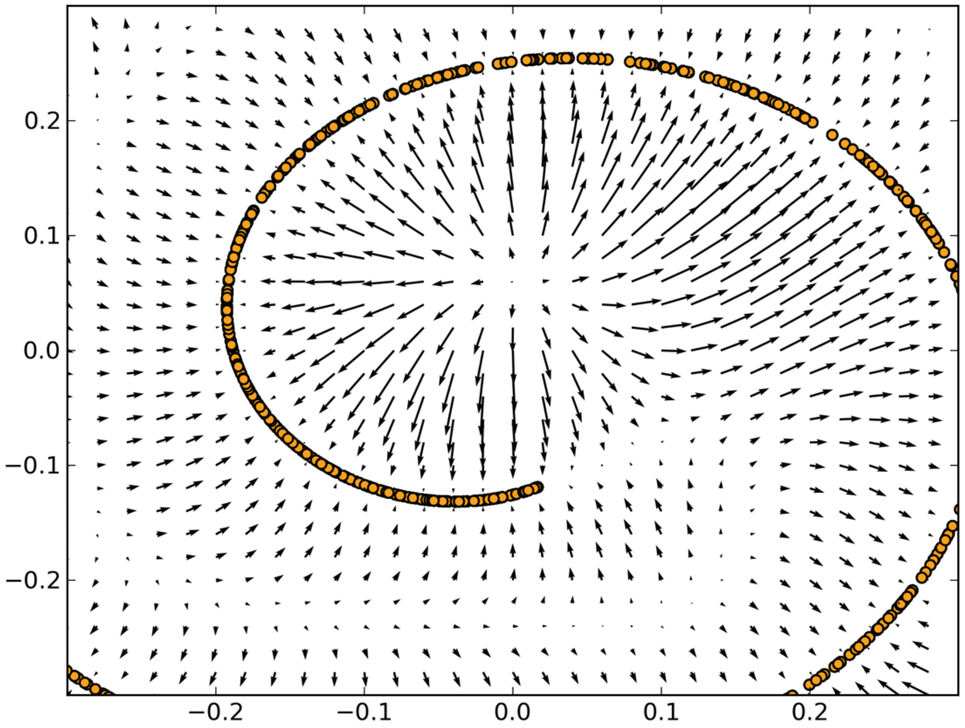
DAEs can learn correct vector fields (reconstruction – noisy input) that point to data manifold (spiral):



zoom out



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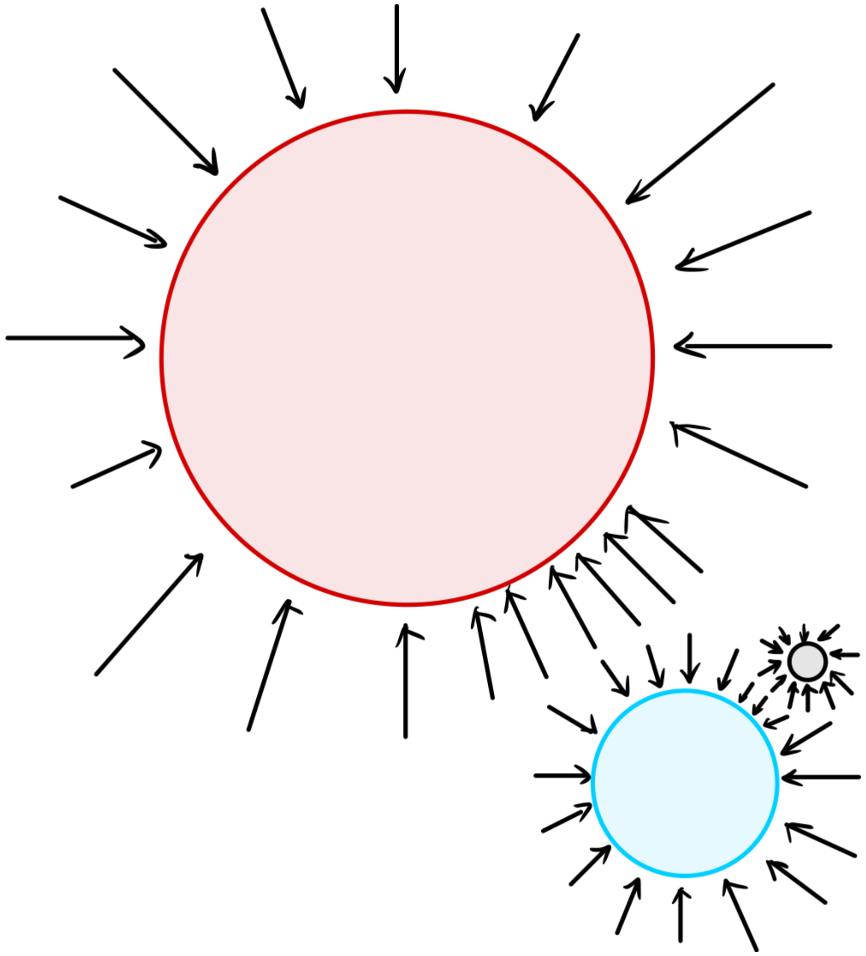

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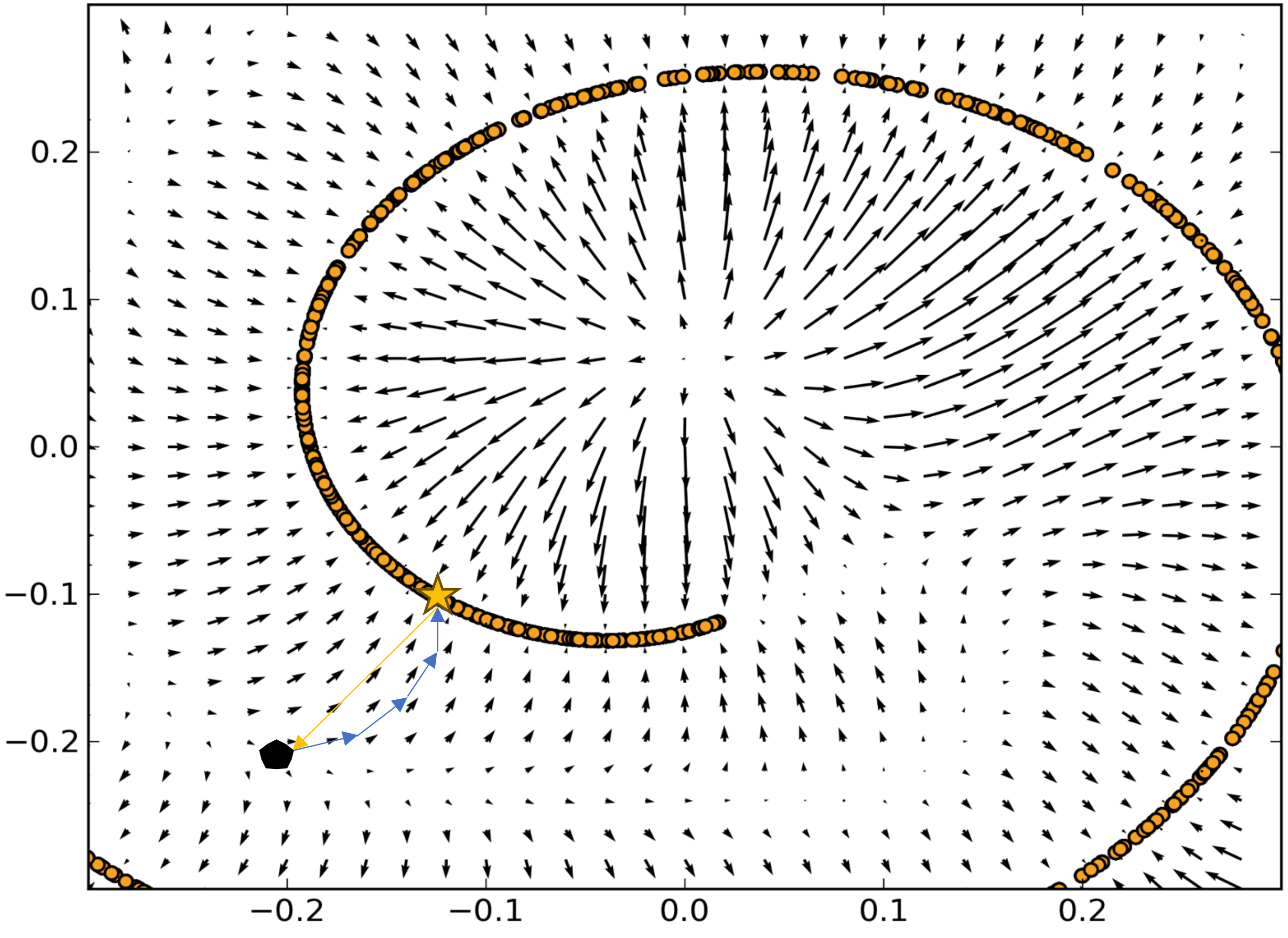

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Reminder:
Please start the course project as soon as possible.

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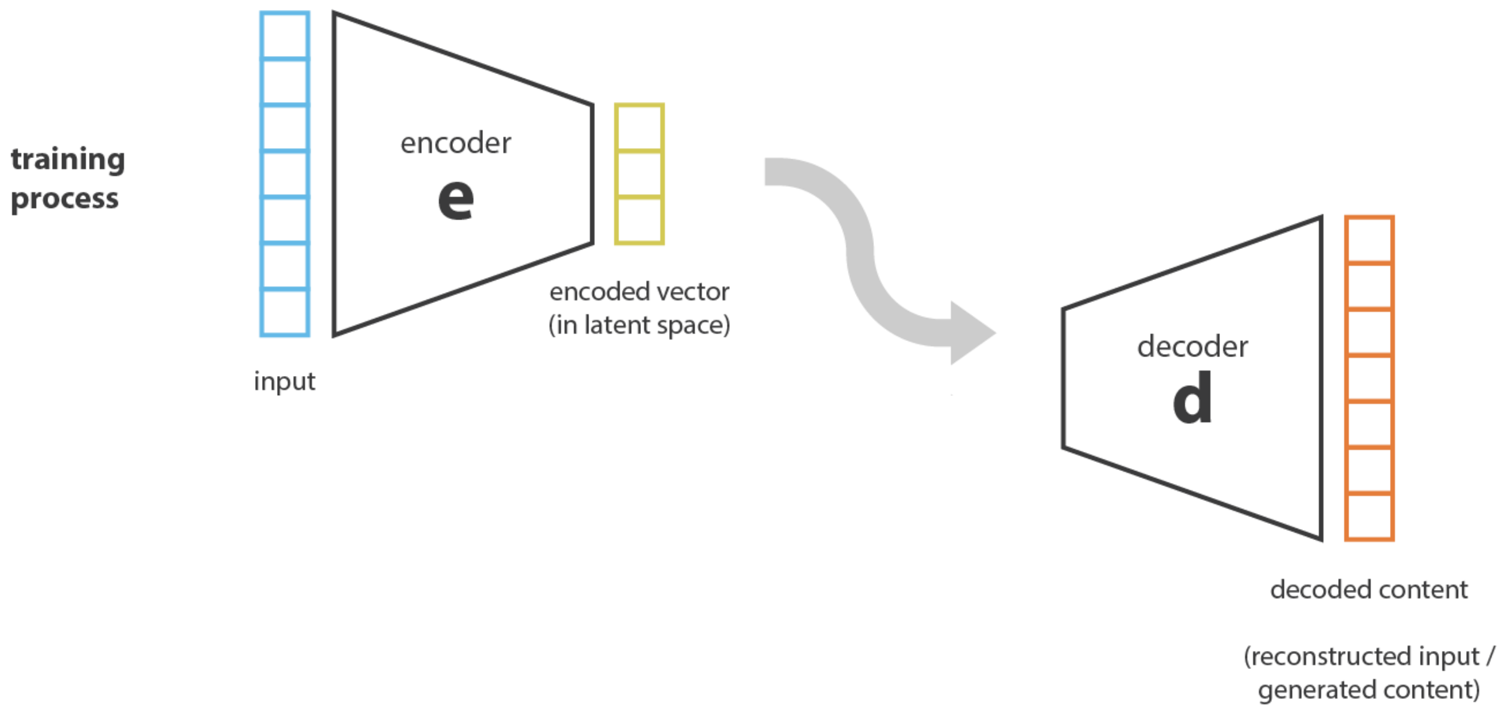

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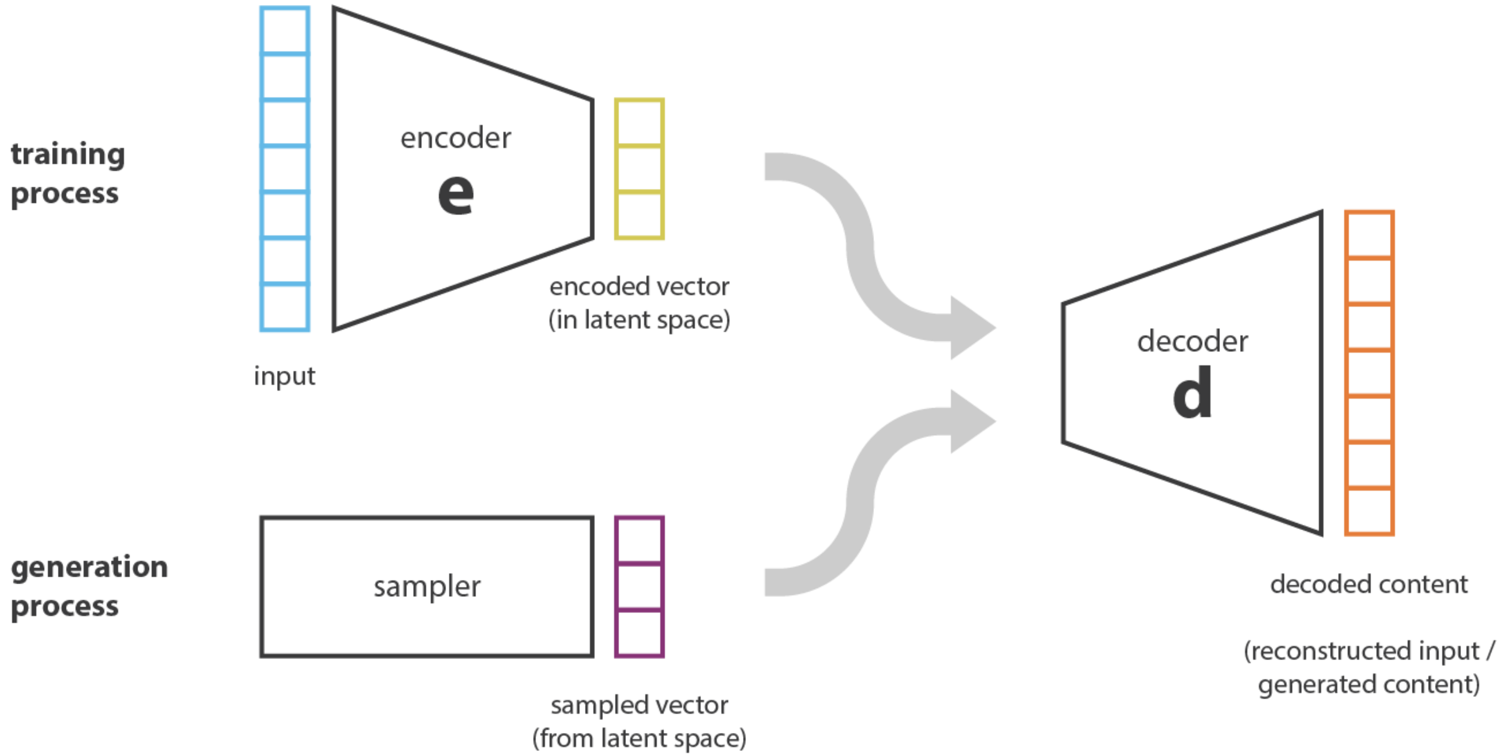

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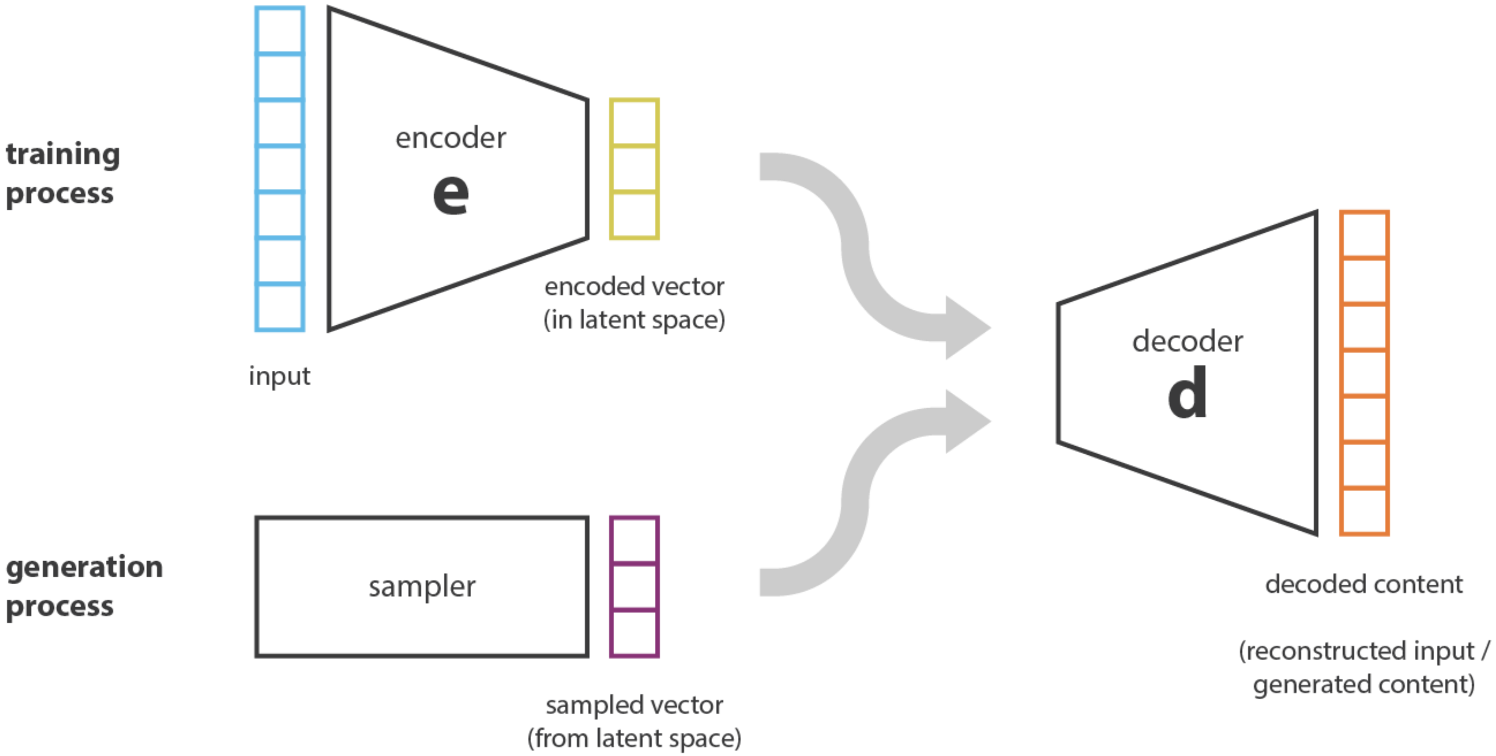

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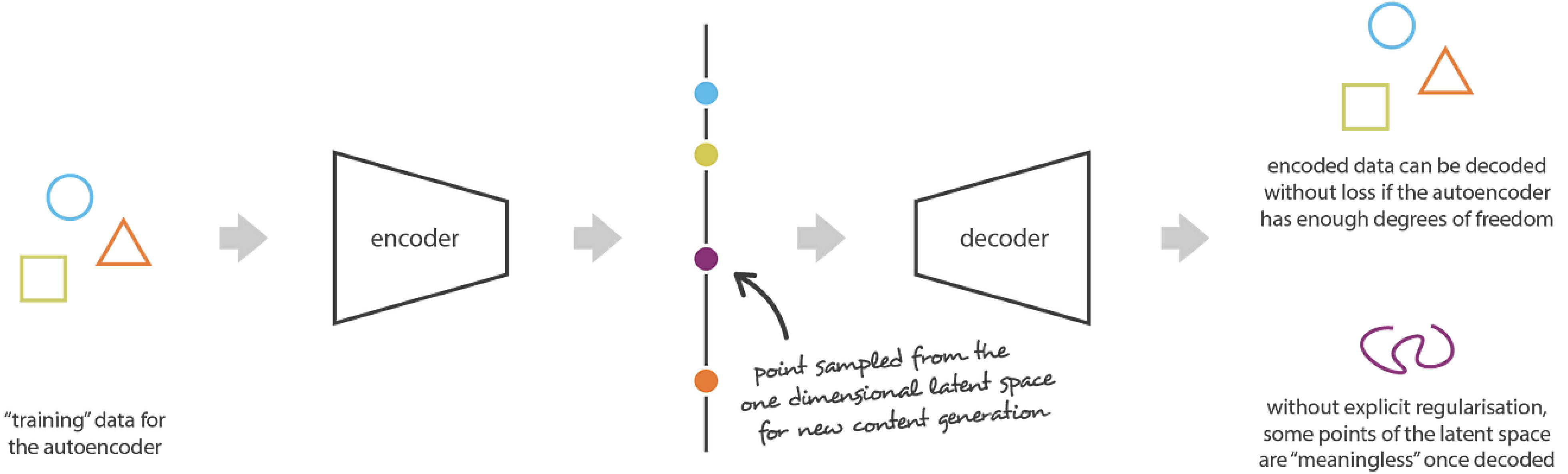
What would happen? *Sampled data could be very bad if sampled latent codes are far off the manifold!*



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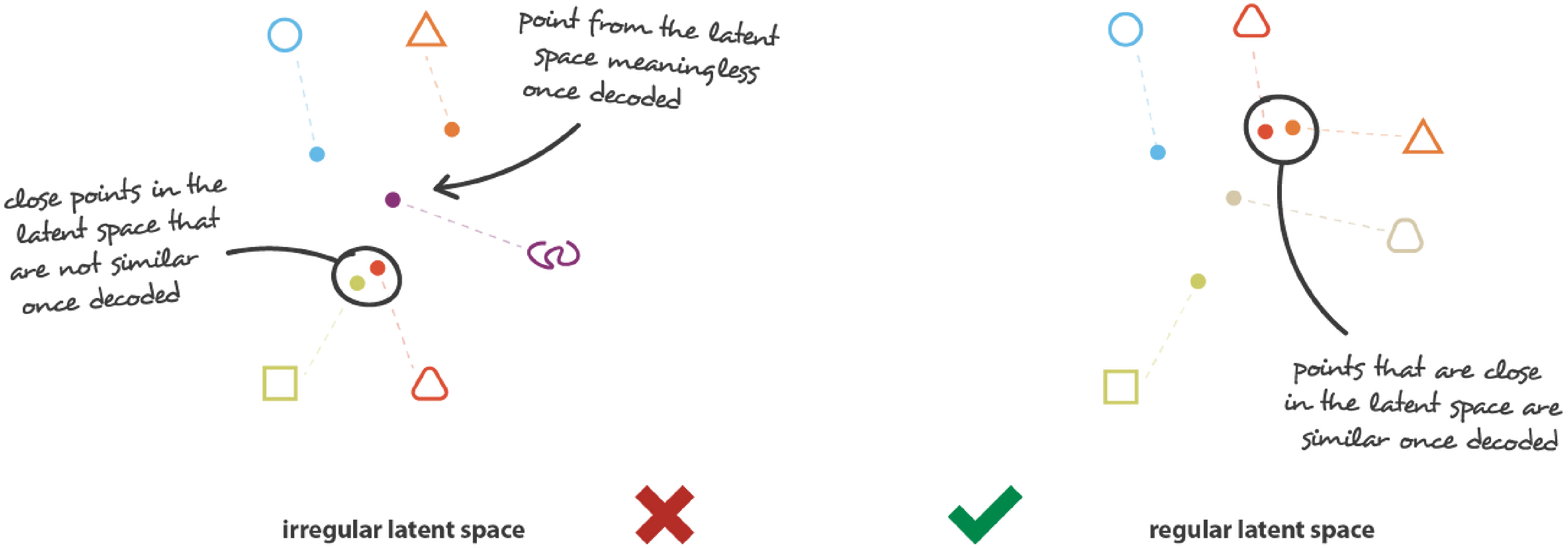

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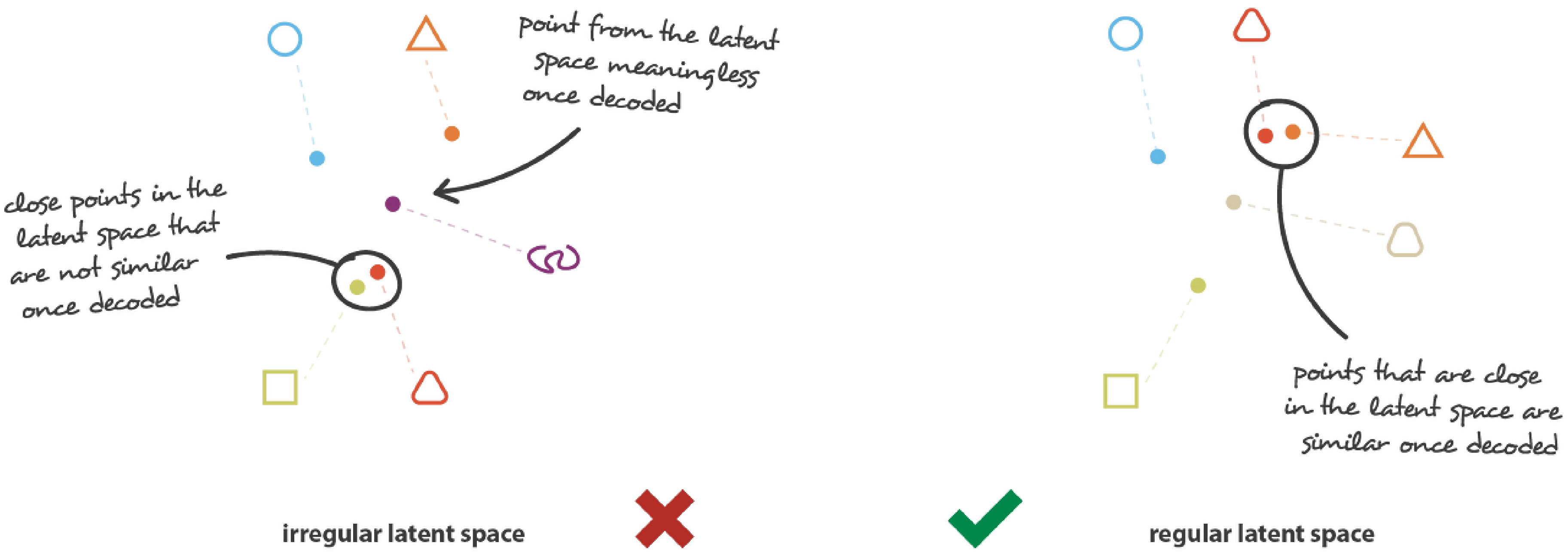

Can AEs learn such latent spaces that are good for reconstruction + generation? Yes, VAEs [7,8]!

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$$\begin{aligned} p_\theta(X) &= \int_Z p_\theta(X, Z) dZ \\ &= \int_Z p_\theta(X|Z) p_\theta(Z) dZ \end{aligned}$$

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Intractable Integration!

Evidence Lower Bound (ELBO)

Variational Approximation

$$\begin{aligned}\log p_\theta(X) &= \log\left(\frac{p_\theta(X,Z)}{p_\theta(Z|X)}\right)\\ &= \log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\frac{q_\phi(Z|X)}{p_\theta(Z|X)}\right)\end{aligned}$$

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Variational Approximation

$$\log p_\theta(X) = \log\left(\frac{p_\theta(X,Z)}{p_\theta(Z|X)}\right)$$
$$= \log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\frac{q_\phi(Z|X)}{p_\theta(Z|X)}\right)$$

Integrating from both sides:

$$\log p_\theta(X) = \int q_\phi(Z|X)\log p_\theta(X)dZ$$
$$= \int q_\phi(Z|X)\log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\frac{q_\phi(Z|X)}{p_\theta(Z|X)}\right)dZ$$
$$= \int q_\phi(Z|X)\log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\right)dZ + \int q_\phi(Z|X)\log\left(\frac{q_\phi(Z|X)}{p_\theta(Z|X)}\right)dZ$$
$$= \mathbb{E}_{q_\phi(Z|X)}\left[\log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\right)\right] + \mathrm{KL}\left(q_\phi(Z|X)\|p_\theta(Z|X)\right)$$

Evidence Lower Bound (ELBO)

Variational Approximation

$$\log p_\theta(X) = \log\left(\frac{p_\theta(X,Z)}{p_\theta(Z|X)}\right)$$
$$= \log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\frac{q_\phi(Z|X)}{p_\theta(Z|X)}\right)$$

Integrating from both sides:

$$\log p_\theta(X) = \int q_\phi(Z|X)\log p_\theta(X)dZ$$
$$= \int q_\phi(Z|X)\log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\frac{q_\phi(Z|X)}{p_\theta(Z|X)}\right)dZ$$
$$= \int q_\phi(Z|X)\log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\right)dZ + \int q_\phi(Z|X)\log\left(\frac{q_\phi(Z|X)}{p_\theta(Z|X)}\right)dZ$$
$$= \underbrace{\mathbb{E}_{q_\phi(Z|X)}\left[\log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\right)\right]}_{\text{Evidence Lower Bound (ELBO)}} + \underbrace{\mathrm{KL}\left(q_\phi(Z|X)\|p_\theta(Z|X)\right)}_{\text{Kullback-Leibler (KL) Divergence}}$$

Evidence Lower Bound (ELBO)

Variational Approximation

$$\log p_\theta(X) = \log\left(\frac{p_\theta(X,Z)}{p_\theta(Z|X)}\right)$$
$$= \log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\frac{q_\phi(Z|X)}{p_\theta(Z|X)}\right)$$

Integrating from both sides:

Why is it a lower bound?

$$\log p_\theta(X) = \int q_\phi(Z|X)\log p_\theta(X)dZ$$
$$= \int q_\phi(Z|X)\log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\frac{q_\phi(Z|X)}{p_\theta(Z|X)}\right)dZ$$
$$= \int q_\phi(Z|X)\log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\right)dZ + \int q_\phi(Z|X)\log\left(\frac{q_\phi(Z|X)}{p_\theta(Z|X)}\right)dZ$$
$$= \underbrace{\mathbb{E}_{q_\phi(Z|X)}\left[\log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\right)\right]}_{\text{Evidence Lower Bound (ELBO)}} + \underbrace{\mathrm{KL}\left(q_\phi(Z|X)\|p_\theta(Z|X)\right)}_{\text{Kullback-Leibler (KL) Divergence}}$$

Evidence Lower Bound (ELBO) Kullback-Leibler (KL) Divergence

Evidence Lower Bound (ELBO)

Variational Approximation

$$\log p_\theta(X) = \log\left(\frac{p_\theta(X,Z)}{p_\theta(Z|X)}\right)$$
$$= \log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\frac{q_\phi(Z|X)}{p_\theta(Z|X)}\right)$$

Integrating from both sides:

Why is it a lower bound? KL is nonnegative!

$$\log p_\theta(X) = \int q_\phi(Z|X)\log p_\theta(X)dZ$$
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Evidence Lower Bound (ELBO)

Variational Approximation

$$\log p_\theta(X) = \log\left(\frac{p_\theta(X,Z)}{p_\theta(Z|X)}\right)$$
$$= \log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\frac{q_\phi(Z|X)}{p_\theta(Z|X)}\right)$$

Integrating from both sides:

Why is it a lower bound? KL is nonnegative!

Why is it called variational approximation?

$$\log p_\theta(X) = \int q_\phi(Z|X)\log p_\theta(X)dZ$$
$$= \int q_\phi(Z|X)\log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\frac{q_\phi(Z|X)}{p_\theta(Z|X)}\right)dZ$$
$$= \int q_\phi(Z|X)\log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\right)dZ + \int q_\phi(Z|X)\log\left(\frac{q_\phi(Z|X)}{p_\theta(Z|X)}\right)dZ$$
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Evidence Lower Bound (ELBO)

Variational Approximation

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Integrating from both sides:

$$\log p_\theta(X) = \int q_\phi(Z|X)\log p_\theta(X)dZ$$
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Evidence Lower Bound (ELBO) Kullback-Leibler (KL) Divergence

Why is it a lower bound? KL is nonnegative!

Why is it called variational approximation? We choose one distribution (function) from a family to approximate the target!

Evidence Lower Bound (ELBO)

Since true posterior $p_\theta(Z|X)$ is often unknown, KL term is intractable

Evidence Lower Bound (ELBO)

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ELBO:

$$\begin{aligned}
\mathbb{E}_{q_\phi(Z|X)}\left[\log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\right)\right] &= \mathbb{E}_{q_\phi(Z|X)}\left[\log\left(\frac{p_\theta(X|Z)p_\theta(Z)}{q_\phi(Z|X)}\right)\right] \\
&= \mathbb{E}_{q_\phi(Z|X)}\left[\log\left(p_\theta(X|Z)\right)\right] + \mathbb{E}_{q_\phi(Z|X)}\left[\log\left(\frac{p_\theta(Z)}{q_\phi(Z|X)}\right)\right] \\
&= -\mathbb{E}_{q_\phi(Z|X)}\left[-\log\left(p_\theta(X|Z)\right)\right] - \mathrm{KL}\left(q_\phi(Z|X)\|p_\theta(Z)\right)
\end{aligned}$$

Evidence Lower Bound (ELBO)

Since true posterior $p_\theta(Z|X)$ is often unknown, KL term is intractable

ELBO:

$$
\begin{aligned}
\mathbb{E}_{q_\phi(Z|X)}\left[\log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\right)\right] &= \mathbb{E}_{q_\phi(Z|X)}\left[\log\left(\frac{p_\theta(X|Z)p_\theta(Z)}{q_\phi(Z|X)}\right)\right] \\
&= \mathbb{E}_{q_\phi(Z|X)}\left[\log\left(p_\theta(X|Z)\right)\right] + \mathbb{E}_{q_\phi(Z|X)}\left[\log\left(\frac{p_\theta(Z)}{q_\phi(Z|X)}\right)\right] \\
&= \underbrace{-\mathbb{E}_{q_\phi(Z|X)}\left[-\log\left(p_\theta(X|Z)\right)\right]}_{\text{Reconstruction Error/Loss}} - \underbrace{\mathrm{KL}\left(q_\phi(Z|X)\|p_\theta(Z)\right)}_{\text{Regularizer}}
\end{aligned}
$$

Evidence Lower Bound (ELBO)

Since true posterior $p_\theta(Z|X)$ is often unknown, KL term is intractable

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$$
\begin{aligned}
\mathbb{E}_{q_\phi(Z|X)}\left[\log\left(\frac{p_\theta(X,Z)}{q_\phi(Z|X)}\right)\right] &= \mathbb{E}_{q_\phi(Z|X)}\left[\log\left(\frac{p_\theta(X|Z)p_\theta(Z)}{q_\phi(Z|X)}\right)\right] \\
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&= \underbrace{-\mathbb{E}_{q_\phi(Z|X)}\left[-\log\left(p_\theta(X|Z)\right)\right]}_{\text{Reconstruction Error/Loss}} - \underbrace{\mathrm{KL}\left(q_\phi(Z|X)\|p_\theta(Z)\right)}_{\text{Regularizer}}
\end{aligned}
$$

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  - Motivation & Overview
  - Linear Autoencoders & PCA
  - Deep Autoencoders
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- Variational Autoencoders
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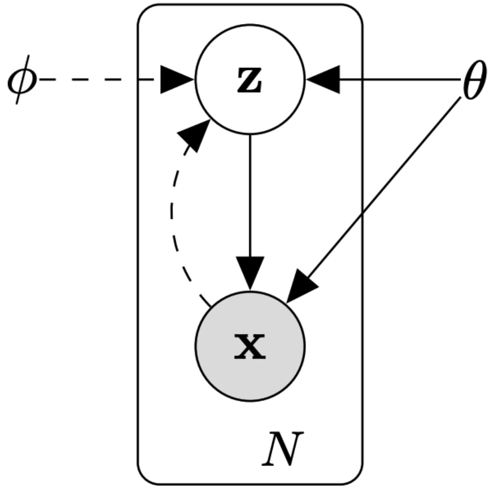
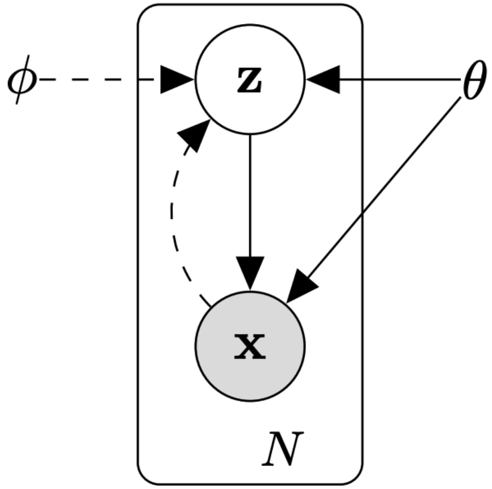
  - Evidence Lower Bound (ELBO)
  - **Models**
  - Amortized Inference
  - Reparameterization Trick

Variational Autoencoders



Encoder:	$q_\phi(Z\|X)$
Decoder:	$p_\theta(X\|Z)$
Prior:	$p_\theta(Z)$

Variational Autoencoders



Since we typically use continuous latent variable Z, Gaussian distribution is a natural choice for the encoder:

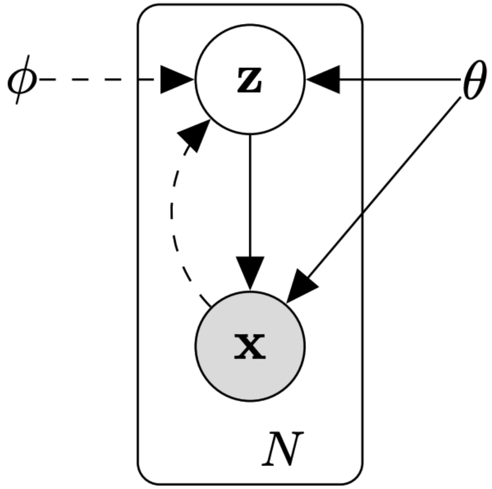
$$q_\phi(Z|X) = \mathcal{N}(Z|\mu, \sigma^2 I)$$
$$\mu = \text{EncoderNetwork}_\phi(X)$$
$$\log \sigma^2 = \text{EncoderNetwork}_\phi(X)$$

Encoder: $q_\phi(Z|X)$

Decoder: $p_\theta(X|Z)$

Prior: $p_\theta(Z)$

Variational Autoencoders



Encoder: $q_\phi(Z|X)$

Decoder: $p_\theta(X|Z)$

Prior: $p_\theta(Z)$

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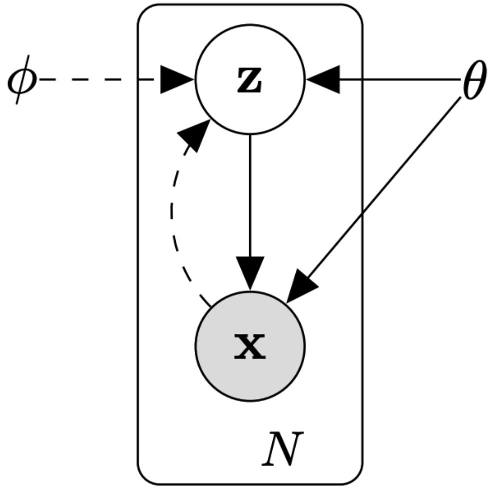
$$q_\phi(Z|X) = \mathcal{N}(Z|\mu, \sigma^2 I)$$
$$\mu = \text{EncoderNetwork}_\phi(X)$$
$$\log \sigma^2 = \text{EncoderNetwork}_\phi(X)$$

Similarly, Gaussian distribution is often adopted for the decoder:

$$p_\theta(X|Z) = \mathcal{N}(X|\tilde{\mu}, \tilde{\sigma}^2 I)$$
$$\tilde{\mu} = \text{DecoderNetwork}_\theta(Z)$$
$$\log \tilde{\sigma}^2 = \text{DecoderNetwork}_\theta(Z)$$

Variational Autoencoders



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$$\mu = \text{EncoderNetwork}_\phi(X)$$
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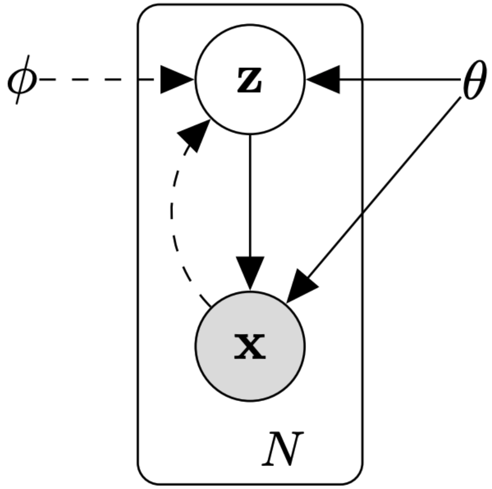
$$\tilde{\mu} = \text{DecoderNetwork}_\theta(Z)$$
$$\log \tilde{\sigma}^2 = \text{DecoderNetwork}_\theta(Z)$$

We often fix the prior as, e.g., standard Normal $p_\theta(Z) = \mathcal{N}(Z|\mathbf{0}, I)$

Variational Autoencoders



Encoder: $q_\phi(Z|X)$

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Illustration of VAEs:

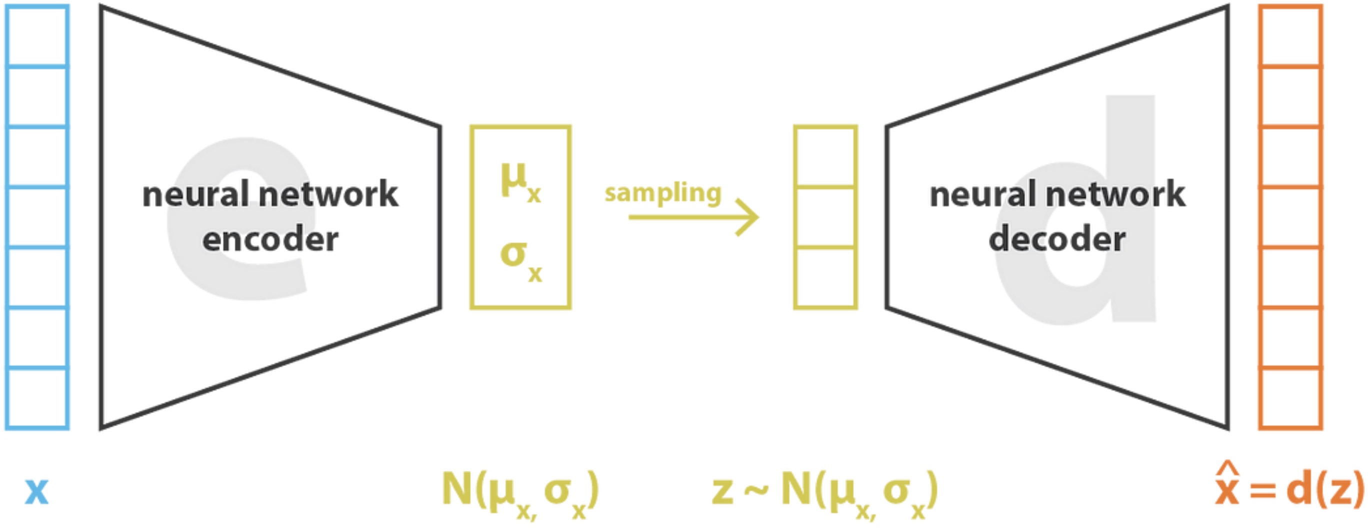

Image Credit: [6]

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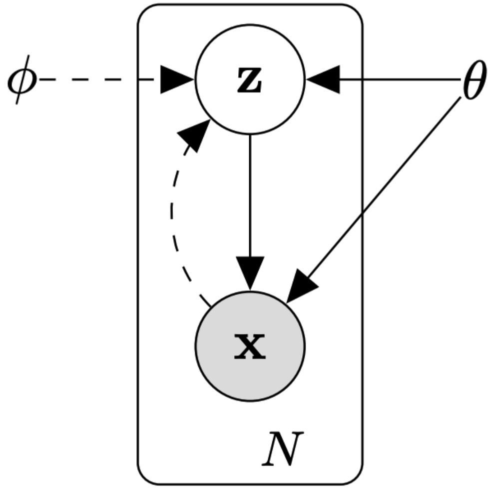
  - Models
  - **Amortized Inference**
  - Reparameterization Trick

Amortized Variational Inference



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$$\mu = \text{EncoderNetwork}_\phi(X)$$
$$\log \sigma^2 = \text{EncoderNetwork}_\phi(X)$$

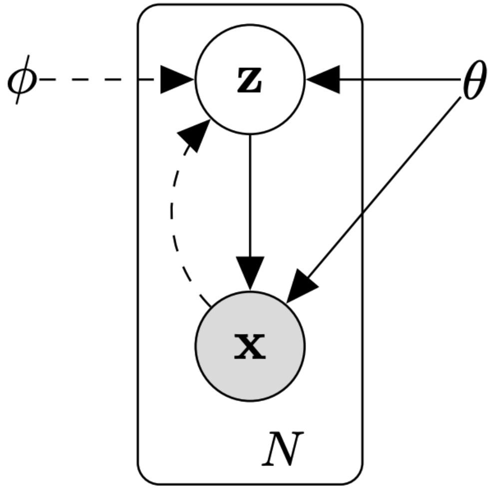
Encoder is amortized: every X shares the same set of parameters ϕ

Encoder: $q_\phi(Z|X)$

Decoder: $p_\theta(X|Z)$

Prior: $p_\theta(Z)$

Amortized Variational Inference



Encoder: $q_\phi(Z|X)$

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Prior: $p_\theta(Z)$

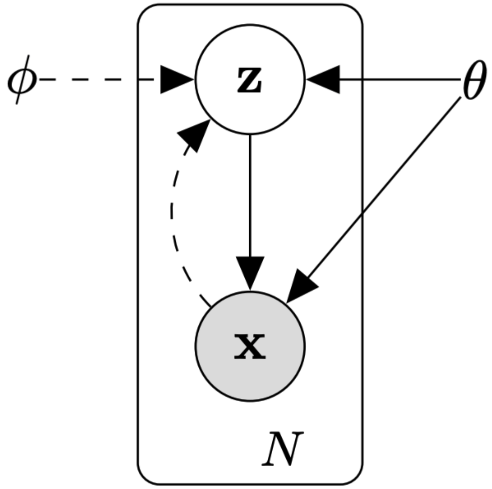
Since we typically use continuous latent variable Z, Gaussian distribution is a natural choice for the encoder:

$$\begin{aligned} q_\phi(Z|X) &= \mathcal{N}(Z|\mu, \sigma^2 I) \\ \mu &= \text{EncoderNetwork}_\phi(X) \\ \log \sigma^2 &= \text{EncoderNetwork}_\phi(X) \end{aligned}$$

Encoder is amortized: every X shares the same set of parameters ϕ

We thus only need to optimize ELBO over one set of parameters ϕ, whereas in traditional variational inference (VI) one needs to find the optimal variational distribution per X

Amortized Variational Inference



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Encoder is amortized: every X shares the same set of parameters ϕ

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Different X still have different encoder distributions $q_\phi(Z|X)$

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Reparameterization Trick

Negative ELBO: $\mathcal{L}(\phi, \theta) = \underbrace{\mathbb{E}_{q_\phi(Z|X)}\left[-\log\left(p_\theta(X|Z)\right)\right]}_{\text{Reconstruction Error/Loss}} + \underbrace{\mathrm{KL}\left(q_\phi(Z|X) \| p_\theta(Z)\right)}_{\text{Regularizer}}$

We want to minimize negative ELBO w.r.t. encoder parameters ϕ and decoder parameters θ

Reparameterization Trick

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We want to minimize negative ELBO w.r.t. encoder parameters ϕ and decoder parameters θ

The expectation in reconstruction loss is intractable and often approximated by Monte Carlo estimation

Reparameterization Trick

Negative ELBO: $$\mathcal{L}(\phi, \theta) = \underbrace{\mathbb{E}_{q_\phi(Z|X)} \left[- \log \left(p_\theta(X|Z) \right) \right]}_{\text{Reconstruction Error/Loss}} + \underbrace{\mathrm{KL} \left(q_\phi(Z|X) \| p_\theta(Z) \right)}_{\text{Regularizer}}$$

We want to minimize negative ELBO w.r.t. encoder parameters ϕ and decoder parameters θ

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Once we draw samples of Z, we can get the Monte Carlo gradient of reconstruction loss w.r.t. θ via backpropagation

Reparameterization Trick

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We want to minimize negative ELBO w.r.t. encoder parameters ϕ and decoder parameters θ

The expectation in reconstruction loss is intractable and often approximated by Monte Carlo estimation

Once we draw samples of Z, we can get the Monte Carlo gradient of reconstruction loss w.r.t. θ via backpropagation

We will use *reparameterization trick* to equivalently rewrite the expectation in reconstruction loss so that the Monte Carlo gradient w.r.t. ϕ has a lower variance.

Reparameterization Trick

For any function f, we have

$$
\begin{aligned}
\mathbb{E}_{\mathcal{N}(Z|\mu,\sigma^2 I)}[f(Z)] &= \int \frac{1}{\sqrt{(2\pi)^m}\prod_i \sigma_i} \exp(-\frac{1}{2}\left\|\frac{Z-\mu}{\sigma}\right\|^2) f(Z)dZ \\
&= \int \frac{1}{\sqrt{(2\pi)^m}\prod_i \sigma_i} \exp(-\frac{1}{2}\left\|\frac{\mu+\sigma\epsilon-\mu}{\sigma}\right\|^2) f(\mu+\sigma\epsilon)d(\mu+\sigma\epsilon) \\
&= \int \frac{1}{\sqrt{(2\pi)^m}} \exp(-\frac{1}{2}\|\epsilon\|^2) f(\mu+\sigma\epsilon)d\epsilon \\
&= \mathbb{E}_{\mathcal{N}(\epsilon|0,I)}[f(\mu+\sigma\epsilon)]
\end{aligned}
$$

Change of Variable

Reparameterization Trick

For any function f, we have

$$\begin{aligned}\mathbb{E}_{\mathcal{N}(Z|\mu,\sigma^2 I)}\left[f(Z)\right] &= \int \frac{1}{\sqrt{(2\pi)^m}\prod_i \sigma_i} \exp(-\frac{1}{2}\left\|\frac{Z-\mu}{\sigma}\right\|^2) f(Z) dZ \\ &= \int \frac{1}{\sqrt{(2\pi)^m}\prod_i \sigma_i} \exp(-\frac{1}{2}\left\|\frac{\mu+\sigma\epsilon-\mu}{\sigma}\right\|^2) f(\mu+\sigma\epsilon) d\left(\mu+\sigma\epsilon\right) \\ &= \int \frac{1}{\sqrt{(2\pi)^m}} \exp(-\frac{1}{2}\left\|\epsilon\right\|^2) f(\mu+\sigma\epsilon) d\epsilon \\ &= \mathbb{E}_{\mathcal{N}(\epsilon|0,I)}\left[f(\mu+\sigma\epsilon)\right]\end{aligned}$$

Change of Variable

Therefore,

$$\begin{aligned}\mathcal{L}(\phi,\theta) &= \mathbb{E}_{q_\phi(Z|X)}\left[-\log\left(p_\theta(X|Z)\right)\right] + \mathrm{KL}\left(q_\phi(Z|X)\|p_\theta(Z)\right) \\ &= \mathbb{E}_{\mathcal{N}(\epsilon|0,I)}\left[-\log\left(p_\theta(X|\mu_\phi(X)+\sigma_\phi(X)\epsilon)\right)\right] + \mathrm{KL}\left(q_\phi(Z|X)\|p_\theta(Z)\right)\end{aligned}$$

Reparameterization Trick

In original VAE,

$$q_\phi(Z|X) = \mathcal{N}(Z|\mu_\phi(X), \sigma_\phi(X)^2 I)$$
$$p_\theta(Z) = \mathcal{N}(X|0, I)$$

Reparameterization Trick

In original VAE,

$$q_\phi(Z|X) = \mathcal{N}(Z|\mu_\phi(X), \sigma_\phi(X)^2 I)$$
$$p_\theta(Z) = \mathcal{N}(X|0, I)$$

Using Gaussian integrals, we have

$$\mathrm{KL}\left(q_\phi(Z|X) \| p_\theta(Z)\right) = \frac{1}{2}\left(\mu_\phi(X)^\top \mu_\phi(X) + \sigma_\phi(X)^\top \sigma_\phi(X)\right) - \frac{1}{2}\sum_{i=1}^{m} \log \sigma_i^2 - \frac{m}{2}$$

where

$$\sigma_\phi(X) = [\sigma_1, \sigma_2, \cdots, \sigma_m]^\top$$

Reparameterization Trick

Therefore, in original VAE, we have

$$\begin{aligned}\mathcal{L}(\phi,\theta) = \ & \mathbb{E}_{\mathcal{N}(\epsilon|0,I)}\left[-\log\left(p_\theta(X|\mu_\phi(X)+\sigma_\phi(X)\epsilon)\right)\right] \\ & + \frac{1}{2}\left(\mu_\phi(X)^\top\mu_\phi(X)+\sigma_\phi(X)^\top\sigma_\phi(X)\right) - \frac{1}{2}\sum_{i=1}^{m}\log\sigma_i^2 - \frac{m}{2}\end{aligned}$$

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We only need *reparameterization trick* and *Monte Carlo estimation* in the first term

$$\begin{aligned}\mathcal{L}(\phi,\theta) \approx \; & -\sum_{i=1,\epsilon_i\sim\mathcal{N}(\epsilon|0,I)}^{N}\log\left(p_\theta(X|\mu_\phi(X)+\sigma_\phi(X)\epsilon_i)\right) \\ & + \frac{1}{2}\left(\mu_\phi(X)^\top\mu_\phi(X)+\sigma_\phi(X)^\top\sigma_\phi(X)\right) - \frac{1}{2}\sum_{i=1}^{m}\log\sigma_i^2 - \frac{m}{2}\end{aligned}$$

Reparameterization Trick

Therefore, in original VAE, we have

$$\begin{aligned}\mathcal{L}(\phi,\theta) = \ & \mathbb{E}_{\mathcal{N}(\epsilon|0,I)}\left[-\log\left(p_\theta(X|\mu_\phi(X)+\sigma_\phi(X)\epsilon)\right)\right] \\ & + \frac{1}{2}\left(\mu_\phi(X)^\top\mu_\phi(X)+\sigma_\phi(X)^\top\sigma_\phi(X)\right) - \frac{1}{2}\sum_{i=1}^{m}\log\sigma_i^2 - \frac{m}{2}\end{aligned}$$

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Now we can get the gradient directly!

Reparameterization Trick

In the illustration of VAEs, the latent variable is *reparameterized* as below:

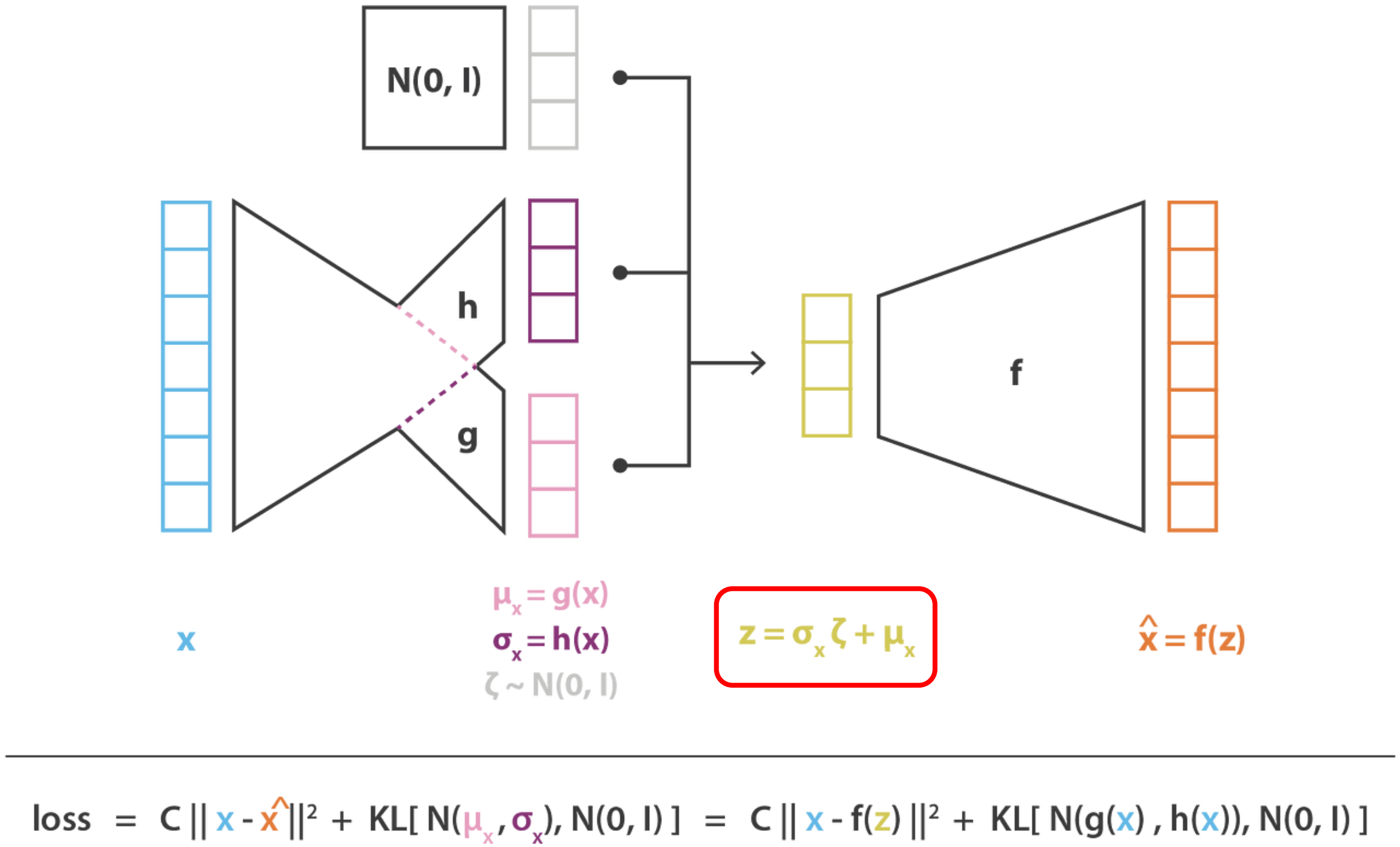
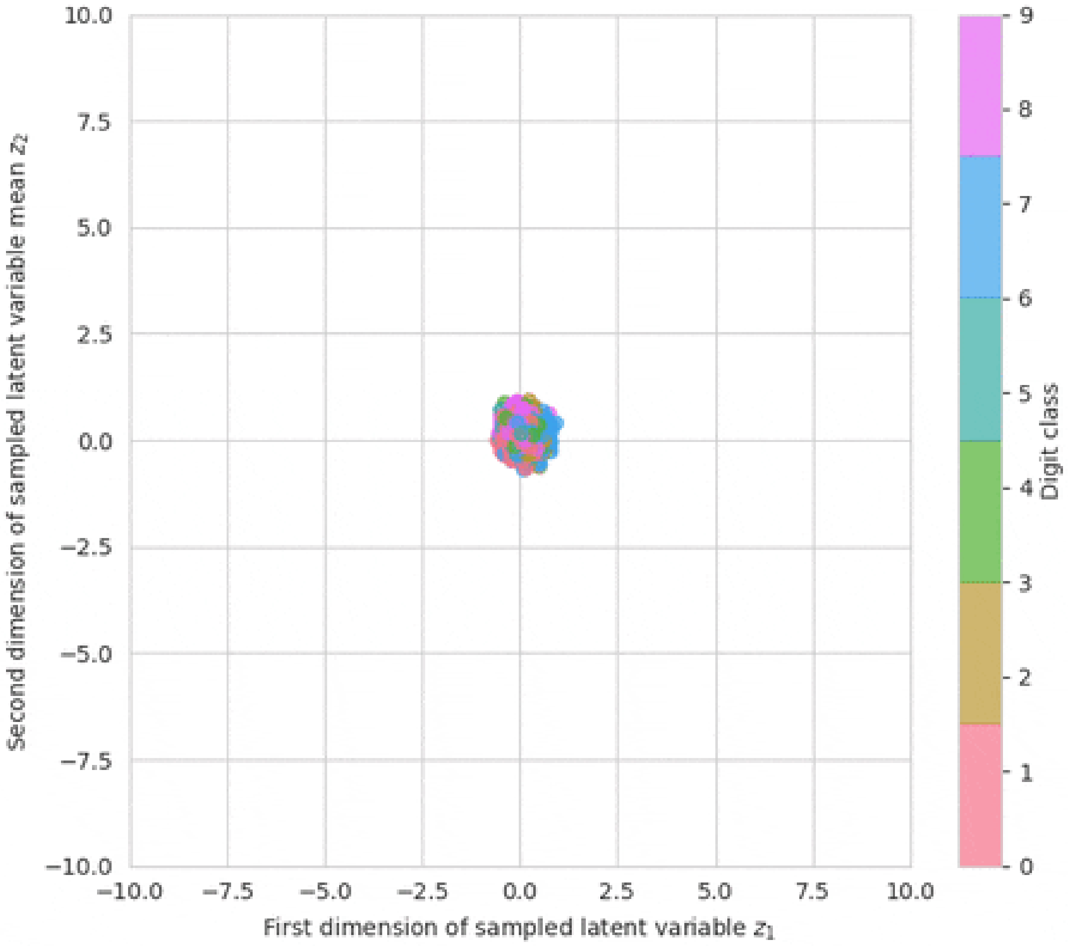


Image Credit: [6]

VAEs on MNIST

Visualize $Z \sim q_\phi(Z|X)$ during training:



Visualize $X \sim p_\theta(X|Z)$ during training:

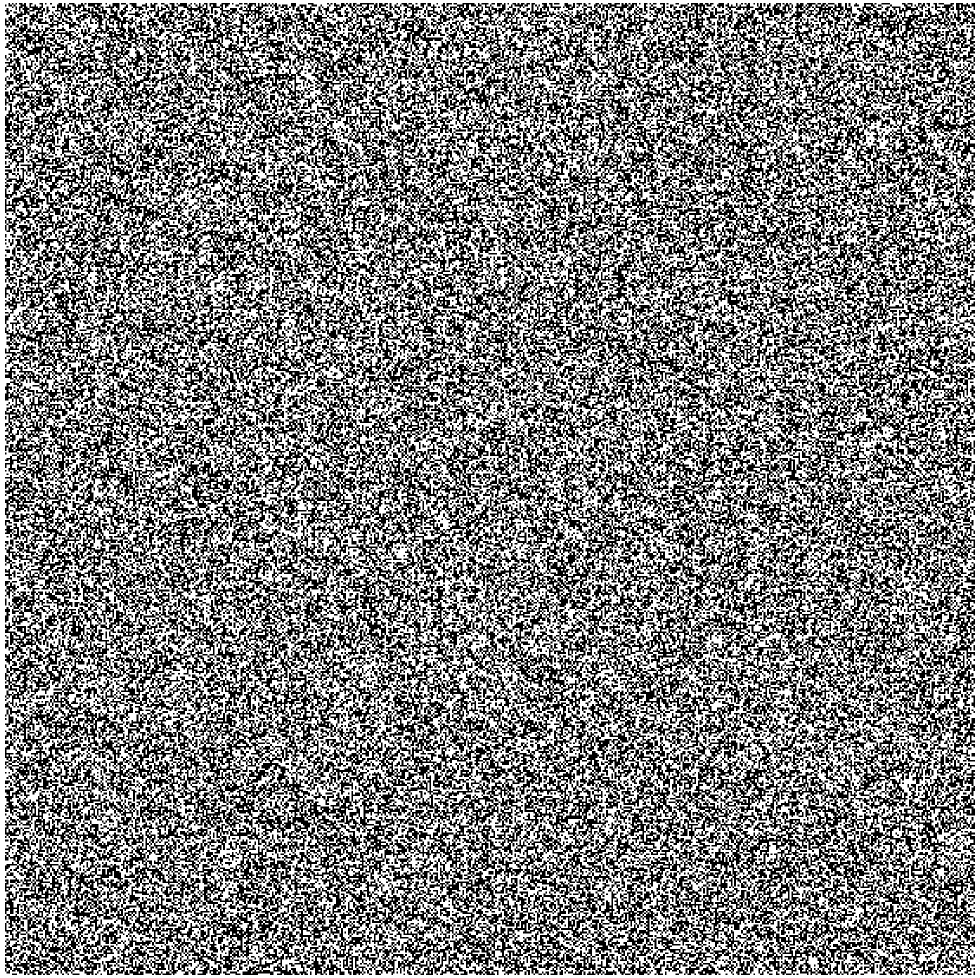


Image Credit: [9]

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Questions?