EECE 517F: Advanced Topics in Deep Learning **Spring 2026**

Lecture 3: Steerable Kernels

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3.1 Steerable Kernels and Basis Functions

Definition of Steerability: Let G be a group acting on a domain X (e.g., $X = \mathbb{R}^2$) via $\mathbf{x} \mapsto g \cdot \mathbf{x}$. A kernel/function $k : X \to \mathbb{C}$ is said to be **steerable with respect to** G if all of its transformed copies under the group action lie in a fixed finite-dimensional subspace.

Formally, using the *left regular representation*, we let G act on functions by

$$[L_g k](\mathbf{x}) \;\triangleq\; k(g^{-1} \cdot \mathbf{x}). \tag{3.1}$$

Then k is steerable if there exist basis functions $\{\psi_1, \ldots, \psi_M\} \subset L^2(X)$ such that for every $g \in G$ we can write

$$[L_g k](\mathbf{x}) = \sum_{j=1}^{M} \alpha_j(g)\, \psi_j(\mathbf{x}), \tag{3.2}$$

where the *steering coefficients* $\alpha_j(g)$ depend only on the group element g (not on $\mathbf{x}$).

Special case (SO(2)): taking $X = \mathbb{R}^2$ with the usual rotation action recovers $[L_{g_\theta} k](\mathbf{x}) = k(R_{-\theta}\mathbf{x})$ and coefficients $\alpha_j(g_\theta) = \alpha_j(\theta)$.

3.2 Example: $SO(2)$ and Circular Harmonics Basis

What is $SO(2)$? $SO(2)$ is simply the set of all **planar rotations**. You can think of it as "angles" $\theta \in [0, 2\pi)$ with addition modulo 2π. Each group element $g_\theta \in SO(2)$ corresponds to a rotation matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{3.3}$$

which rotates any point $\mathbf{x} \in \mathbb{R}^2$.

Why polar coordinates are natural When studying rotations, it is convenient to write points in polar coordinates $\mathbf{x} = (r, \phi)$. A rotation by θ keeps the radius r unchanged and only shifts the angle:

$$(r, \phi) \mapsto (r, \phi + \theta). \tag{3.4}$$

Equivalently, under the left regular action (as defined above), rotating a kernel corresponds to the angular shift

$$[L_{g_\theta} k](r, \phi) = k(r, \phi - \theta). \tag{3.5}$$

Circular harmonics: the Fourier basis on a circle For any fixed radius r, the function $\phi \mapsto k(r, \phi)$ is a (square-integrable) function on the circle. So we can expand it in the standard Fourier series basis on S^1:

$$\psi_m(\phi) \triangleq e^{im\phi}, \quad m \in \mathbb{Z}. \tag{3.6}$$

These $e^{im\phi}$ are called **circular harmonics**. Allowing the coefficient to depend on r gives the general form

$$k(r, \phi) = \sum_{m \in \mathbb{Z}} R_m(r)\, e^{im\phi}, \tag{3.7}$$

where $R_m(r)$ are *radial profiles*.

The key intuition: rotation becomes a phase shift If we rotate the kernel by θ, each basis function simply picks up a complex phase:

$$[L_{g_\theta}\psi_m](r,\phi) = \psi_m(r,\phi-\theta) = e^{-im\theta}\,\psi_m(r,\phi). \tag{3.8}$$

So in this basis, "rotate the filter" is the same as "multiply each frequency-m component by $e^{-im\theta}$". These phases $e^{-im\theta}$ are exactly the **steering weights**.

In the next subsection we derive this basis more systematically from the eigenfunction viewpoint.

3.2.1 Derivation: The Harmonic Basis

We now derive the circular-harmonic basis from the steerability requirement. The key idea is to choose a basis in which the group action (rotation) becomes as simple as possible on the coefficients.

Setup (recall) For $SO(2)$ acting on $\mathbb{R}^2$, the left regular action is

$$[L_{g_\theta}k](\mathbf{x}) = k(R_{-\theta}\mathbf{x}), \tag{3.9}$$

or equivalently in polar coordinates,

$$[L_{g_\theta}k](r,\phi) = k(r,\phi-\theta). \tag{3.10}$$

Thus for each fixed radius r, the slice $k(r,\cdot)$ is a function on the circle S^1.

Because the complex exponentials $\{e^{im\phi}\}_{m\in\mathbb{Z}}$ form a **complete orthogonal basis**[1] for $L^2(S^1)$, any square-integrable kernel can be expanded as a Fourier series along the angular axis:

$$k(r,\phi) = \sum_{m\in\mathbb{Z}} R_m(r)e^{im\phi}, \tag{3.11}$$

where coefficients $R_m(r)$ are **radial functions**.

Using L_g to derive the Basis A kernel is **steerable** if the action of L_{g_θ} preserves the structure of the basis functions. The most efficient way to achieve this is to choose basis functions ψ that are **eigenfunctions** of the operator L_{g_θ}.

Let's apply L_{g_θ} to a single term $\psi_m(r,\phi) = R_m(r)e^{im\phi}$:

$$\begin{aligned}
[L_{g_\theta}\psi_m](r,\phi) &= \psi_m(r,\phi-\theta) && \text{(Definition of } L_g\text{)} && (3.12)\\
&= R_m(r)e^{im(\phi-\theta)} && && (3.13)\\
&= R_m(r)e^{im\phi}e^{-im\theta} && && (3.14)\\
&= e^{-im\theta}\psi_m(r,\phi) && \text{(Eigenvalue Property)} && (3.15)
\end{aligned}$$

This result shows why we use representations:

- The operator L_{g_θ} has been reduced to multiplication by a scalar $\rho(\theta) = e^{-im\theta}$.
- This scalar $e^{-im\theta}$ is exactly the representation of $SO(2)$ for frequency m.

The Benefits of the Basis By expanding the kernel in this specific basis, the operation "Rotate the Filter" ($L_{g_\theta}k$) becomes "Phase Shift the Coefficients" ($e^{-im\theta}R_m$).

$$L_{g_\theta}\left(k(r,\phi)\right) = L_{g_\theta}\left(\sum_m R_m(r)e^{im\phi}\right) = \sum_m (e^{-im\theta}R_m(r))e^{im\phi} \tag{3.16}$$

This allows the network to compute the response to *any* rotated version of the filter analytically, simply by multiplying the output features by $e^{-im\theta}$.

[1] See appendix for more explanation on this point.

Appendix I: Circular Harmonics on S^1 (Derivation, Orthogonality, Completeness)

This appendix gives a self-contained justification for using the basis functions $\{e^{im\phi}\}_{m\in\mathbb{Z}}$ ("circular harmonics") as the Fourier basis on the circle S^1.

References

For rigorous treatments of Fourier series on S^1 and completeness in L^2, see:

1. **Stein, E. M. and Shakarchi, R.** (2003). *Fourier Analysis: An Introduction.* Princeton.
2. **Katznelson, Y.** (2004). *An Introduction to Harmonic Analysis* (3rd ed.). Cambridge.
3. **Folland, G. B.** (1995). *A Course in Abstract Harmonic Analysis.* CRC Press. (Specializes to S^1 as a compact group.)

A.1 Derivation: eigenfunctions of angular shifts

Step 0: Define the shift operator T_θ On the circle S^1 (parameterized by an angle $\phi \in [0, 2\pi)$), rotating the *input angle* by θ corresponds to shifting the argument of the function. We denote this shift operator by T_θ:

$$[T_\theta f](\phi) \triangleq f(\phi - \theta). \tag{3.17}$$

Step 1: T_θ preserves the L^2 inner product (unitarity) Recall the $L^2(S^1)$ inner product

$$\langle f, g\rangle \triangleq \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,\overline{g(\phi)}\,d\phi. \tag{3.18}$$

Then T_θ preserves inner products (hence norms) by a change of variables:

$$\langle T_\theta f, T_\theta g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\phi-\theta)\,\overline{g(\phi-\theta)}\,d\phi \tag{3.19}$$

$$\overset{u=\phi-\theta}{=} \frac{1}{2\pi}\int_0^{2\pi} f(u)\,\overline{g(u)}\,du \tag{3.20}$$

$$= \langle f, g\rangle. \tag{3.21}$$

Therefore T_θ is unitary on $L^2(S^1)$.

Step 2: Eigenfunction assumption ("pure frequencies") In the ordinary Fourier transform, complex exponentials are special because translating the signal only multiplies the coefficient by a phase. Here, we want the analogous property for angular shifts, so we *assume* f has the eigenfunction property: shifting by θ only rescales f. That is, we look for f such that:

$$f(\phi - \theta) = \lambda(\theta) f(\phi) \quad \forall \phi, \theta. \tag{3.22}$$

Step 3: Multiplicativity of the eigenvalue λ The shift operators compose according to angle addition:

$$T_{\theta_1} T_{\theta_2} = T_{\theta_1+\theta_2}. \tag{3.23}$$

Apply both sides to an eigenfunction f. Using (3.22),

$$[T_{\theta_1+\theta_2} f](\phi) = [T_{\theta_1}(T_{\theta_2} f)](\phi) = T_{\theta_1}(\lambda(\theta_2) f)(\phi) = \lambda(\theta_2)[T_{\theta_1} f](\phi) = \lambda(\theta_2)\lambda(\theta_1) f(\phi). \tag{3.24}$$

But also $[T_{\theta_1+\theta_2} f](\phi) = \lambda(\theta_1+\theta_2) f(\phi)$, so

$$\lambda(\theta_1 + \theta_2) = \lambda(\theta_1)\lambda(\theta_2). \tag{3.25}$$

Step 4: Unit-modulus eigenvalues Because T_θ is unitary, it preserves $\|f\|_2$. If f satisfies (3.22), then

$$\|f\|_2 = \|T_\theta f\|_2 = \|\lambda(\theta) f\|_2 = |\lambda(\theta)|\, \|f\|_2, \tag{3.26}$$

so (for nonzero f) we must have $|\lambda(\theta)| = 1$. The (measurable/continuous) solutions are complex phases

$$\lambda(\theta) = e^{-i\mu\theta} \quad \text{for some } \mu \in \mathbb{R}. \tag{3.27}$$

Step 5: Solve for the eigenfunctions and quantize the frequency Plugging back into (3.22) and setting $\theta = \phi$ gives

$$f(0) = e^{-i\mu\phi} f(\phi) \quad \Rightarrow \quad f(\phi) = f(0) e^{i\mu\phi}. \tag{3.28}$$

Finally, functions on the circle are 2π-periodic, so

$$f(\phi + 2\pi) = f(\phi) \Rightarrow e^{i\mu 2\pi} = 1 \Rightarrow \mu \in \mathbb{Z}. \tag{3.29}$$

Therefore the eigenfunctions are exactly

$$\psi_m(\phi) = e^{im\phi}, \quad m \in \mathbb{Z}. \tag{3.30}$$

A.2 Orthogonality in $L^2(S^1)$

Use the standard inner product on $L^2(S^1)$:

$$\langle f, g \rangle \triangleq \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, \overline{g(\phi)}\, d\phi. \tag{3.31}$$

Then for integers m, n,

$$\langle e^{im\phi}, e^{in\phi} \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)\phi} d\phi = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases} \tag{3.32}$$

So $\{e^{im\phi}\}$ is an orthonormal set.

A.3 Completeness in $L^2(S^1)$ (Fourier series theorem)

Completeness means: if $f \in L^2(S^1)$ is orthogonal to every $e^{im\phi}$, then $f = 0$ (in L^2). Equivalently, the linear span of $\{e^{im\phi}\}$ (trigonometric polynomials) is dense in $L^2(S^1)$.

For any $f \in L^2(S^1)$, define its Fourier coefficients

$$\hat{f}(m) \triangleq \langle f, e^{im\phi} \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) e^{-im\phi} d\phi. \tag{3.33}$$

The Fourier series partial sums

$$S_N f(\phi) \triangleq \sum_{m=-N}^{N} \hat{f}(m) e^{im\phi} \tag{3.34}$$

converge to f in L^2 as $N \to \infty$, and we have Parseval's identity

$$\|f\|^2_{L^2(S^1)} = \sum_{m \in \mathbb{Z}} |\hat{f}(m)|^2. \tag{3.35}$$

In particular, if $\hat{f}(m) = 0$ for all $m \in \mathbb{Z}$, then $\|f\|^2_{L^2(S^1)} = 0$, hence $f = 0$. This proves $\{e^{im\phi}\}_{m \in \mathbb{Z}}$ is a complete orthonormal basis of $L^2(S^1)$.

Connection to group representations For S^1 (equivalently $SO(2)$), all irreducible representations are 1D and given by the characters $\rho_m(\theta) = e^{-im\theta}$. The basis functions $e^{im\phi}$ are exactly the corresponding Fourier basis functions on the circle.